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SUPERSONIC FLOW ABOUT AXIALLY SYMMETRICAL DIFFUSERS

BY

JERZY SIELAWA

PREPARED UNDER CONTRACT
N.o AF 49 (638) - 581
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

INSTITUTO TECNOLÓGICO DE AERONÁUTICA
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SUMMARY

This report describes applications of the method of characteristics to supersonic flow about axially symmetrical, open-nosed bodies, such as diffusers. A general procedure is given for the computation of velocity distributions on the external surface of axially symmetrical diffusers of arbitrary shape. The case of straight-line diffusers is developed in more detail, including the case with incidence. A special function is introduced which allows a quick analytical solution of the problem of straight-line diffusers.

ACKNOWLEDGEMENTS

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CONTENTS

1. INTRODUCTION

The main purpose of this paper is the discussion of the axially symmetrical supersonic flow past open nose diffusers (with the exception of the last section, where also the solution for flow with an incidence angle is outlined).

The diffusers are assumed to have a sharp lip (see figure 1) and such shape that the flow is isentropic and may satisfy certain simplifying assumptions discussed in section 2.

In discussing the flow past diffusers only the external one is considered. For this to be justified, perfect matching must be assumed; i.e. there is no "spillage".

The x,y reference system is chosen in such way

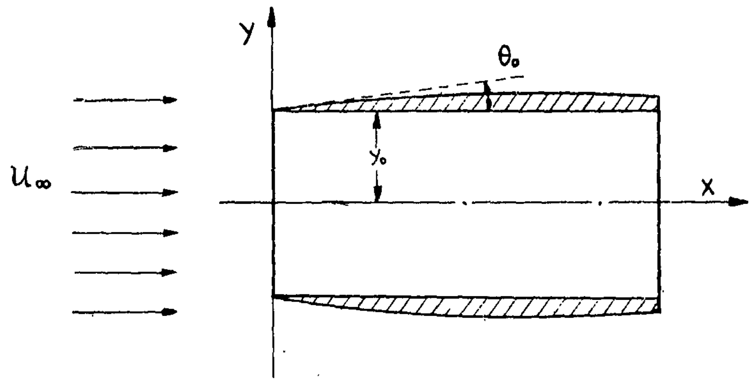


Fig. 1

that the x axis coincides with the symmetry axis, and has the same sense as the undisturbed flow velocity.

The local surface angles, formed by the tangent of the generator of the diffuser outer surface, with the symmetry axis, are called θ .

All the conditions at the nose point are designated with the subscript "o";

e.g. y_o, θ_o, u_o, etc.

Notations and definitions follow those normally used. Particularly

W - represents the value of flow velocity at any point of the flow field.

W_∞ - the value of velocity in the undisturbed region.

U,V - represent the horizontal and vertical components of the velocity $\underline{W}$, respectively.

u,v - represent the horizontal and vertical components of the perturbation velocity.

Due to coincidence of the direction of W_∞ and the $\underline{x}$ axis, it is possible to form the following relations:

$$U_\infty = W_\infty$$

$$V_\infty = 0$$

$$U = U_\infty + u$$

$$V = v$$

The symbol β denotes cot μ , where μ is the Mach angle; $\underline{M}$-denotes the Mach number ($\beta = \sqrt{M^2-1}$) .By "Mach lines" we shall understand the straight lines, tangent to the characteristic lines. Both lines (Mach and characteristic) are, of course, identical in the undisturbed flow region.

In later sections the "straight line diffuser" will be separately discussed. By this name we shall understand those diffusers whose outer surface forms a straight line in an axial section (see figure 7).

Here is a short outline of the paper:

In sections 2 to 8 the simplified characteristic equations according to the Sauer-Heinz method are adapted to diffusers.

Nets of Mach lines and equations for u and v are established for each point of net which allow the calculation of these velocities from known data at former points of net. Next, by means of algebrical transformations, the v - components of pert ~~otion~~ velocity are eliminated, and, in special cases, solutions are found for u directly from the geometrical position of the points ("the first Mach line", see section 5).

After adapting the above described method to straight line diffusers (section 9), the possibility is outlined of solving this problem (i.e. of straight line diffusers) by analytical means; this is facilitated by assuming a function $\sigma(x, \theta, \beta)$, defined by equation (16), section 10, and its limit $\sigma(x, \beta)$ for $\theta \to 0$ (section 11).

The next sections are dedicated to finding the numerical values of function $\sigma(x, \beta)$ (section 12) and its analytical form, first on the basis of formerly described methods (section 13), and next, utilizing operational calculus (sections 14-16). There are obtained an exact solution-equation (39) and approximate solutions-equations(22), (40) and (41).

In sections 17-18 the approximate solution of the u-velocity distribution is given, not only for the diffuser surface, but for the entire space around it.

Finally, in the last section, the solution of flow with an incidence angle is outlined.

Graphs, ilustrating all more important results, are enclosed.

Part One - GENERAL METHODS

2. THE DERIVATION OF THE SAUER-HEINZ EQUATION

The considerations of this section will be based on the well known differential equation of the characteristics for an axially symmetrical isentropic flow, namely:

$$\pm d\theta - \frac{dw}{w}\cot\mu + \sin\mu \sin\theta \frac{dl}{y} = 0 \qquad (1)$$

where the top signs refer to the differentiation along the left running (L) - and the bottom - along the right running characteristic lines (R - see figure 2).

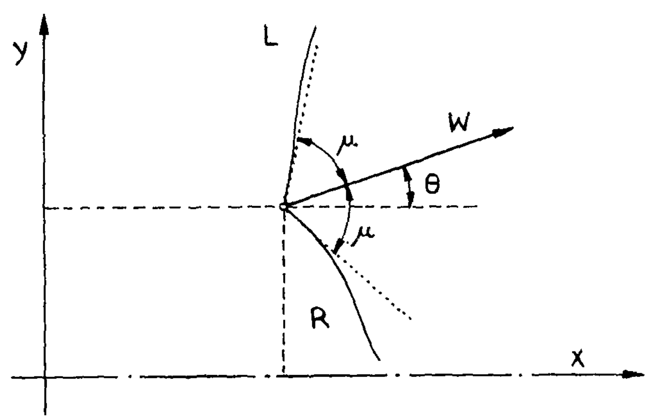


Fig. 2

Replacing the velocity W by its components U and v, and the element dl by the expression $\pm dy/\sin(\mu\pm\theta)$, after some transformations, one gets:

$$dv \mp \cot(\mu \mp \theta) + \frac{\sin^2\mu}{\sin(\mu+\theta)\sin(\mu-\theta)} \frac{v\,dy}{y} = 0 \qquad (1')$$

In order to simplify the above equation, two assumptions will be made; these are approximately equivalent to the linear theory of V. Kármán and Moore. These assumptions are:

(i) - $\mu \gg \theta$, which means that angles θ must be relatively small, and Mach numbers not too great.

(ii) - $\mu \simeq \mu_\infty$, that is, the Mach numbers are almost constant and equal approximately to M_∞ for the undisturbed flow.

The assumption (i) implies, obviously, that:

$$\mu \pm \theta \simeq \mu$$

which enables to present equation (1') in a very simple final form:

$$d(vy) \mp \beta y \, dU = 0 \qquad (2)$$

The assumption (ii), which to a certain point is the consequence of (i), means that all the Mach lines may be treated as mutually parallel straight lines. This fact makes the integration of the equation (2) considerably easier.

3. THE INTEGRATION OF THE SAUER-HEINZ EQUATION

In order to perform the integration of equation (2), let us choose a certain number of arbitrary points on the body surface and draw therefrom left (L) and right running (R) Mach lines, parallel to the Mach lines of the undisturbed flow. This procedure will create a "net" of Mach lines which we shall call a Mach net.

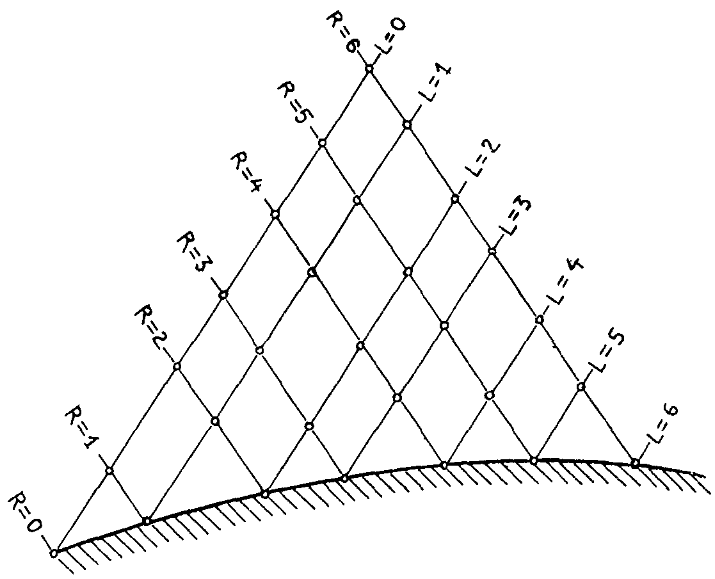


Fig. 3

To facilitate the calculation, let us number the lines of the Mach net in the manner shown in figure 3. Points, which are the intersections of the left Mach line with number L and of the right one with number R, will be designated by (L;R).

Assuming that the distances between neighboring points in the Mach net are small enough, we may write:

$$\int_{(L;R-1)}^{(L;R)} y\,dU \simeq \frac{1}{2}(y_{L;R}+y_{L;R-1})(U_{L;R}-U_{L;R-1}) =$$

$$= \frac{1}{2}(y_{L;R}+y_{L;R-1})(u_{L;R}-u_{L;R-1})$$

and

$$\int_{(L-1;R)}^{(L;R)} y\,dU \simeq \frac{1}{2}(y_{L;R}+y_{L-1;R})(U_{L;R}-U_{L-1;R}) =$$

$$= \frac{1}{2}(y_{L;R}+y_{L-1;R})(u_{L;R}-u_{L-1;R})$$

thus, the result of integrating (2) may be written

$$(vy)_{L;R}-(vy)_{L;R-1} = \frac{1}{2}\beta(y_{L;R}+y_{L;R-1})(u_{L;R}-u_{L;R-1}) \qquad (3)$$

$$(vy)_{L;R}-(vy)_{L-1;R} = -\frac{1}{2}\beta(y_{L;R}+y_{L-1;R})(u_{L;R}-u_{L-1;R}) \qquad (3')$$

the boundary condition is:

$$v_{K;K} = (U_\infty+u)\tan\theta_K \simeq (U_\infty+u)\theta_K \qquad (3'')$$

where θ_K denotes the slope at point $(K;K)$.

The problem of determining flow velocities in the net points will be now discussed in four parts according to the position of the points.

4. THE DIFFUSER NOSE POINT

Let us choose two points on the same Mach line: P on the diffuser surface and Q in the undistrubed region (see figure 4)

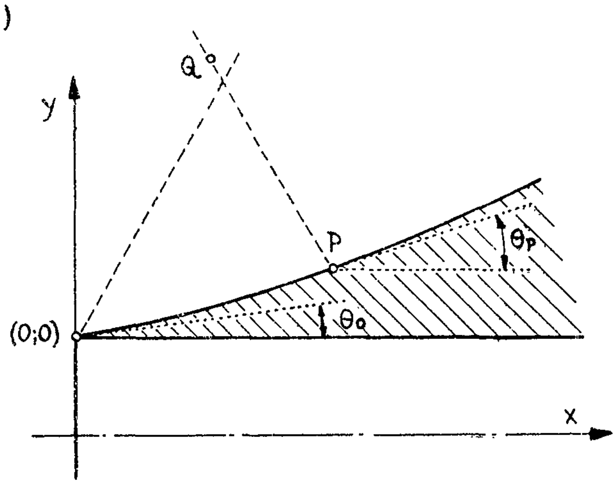


Fig. 4

In order to calculate the velocity u_p it is possible to use equations (3') and (3")

Solving these equations and assuming that $P \to (0;0)$ and $Q \to (0;0)$, one gets immediately:

$$u_0 = -\frac{\theta_0}{\beta+\theta_0}U_\infty \simeq -\frac{\theta_0}{\beta}U_\infty \qquad (4)$$

where the subscript "0" refers to the point (0;0). The equation (4) is, of course, equal to the result for a two dimensional flow, deflected through an angle θ_0. Obviously, this must have been expected, considering that the problem in the close neighborhood of the nose point is indeed two dimensional.

5. POINTS ON THE FIRST MACH LINE (L=0)

In order to solve the problem for the first Mach line, i.e. the Mach line with L=0, let us choose on it two points: (0;R) and (0;R-1) (see figure 5)

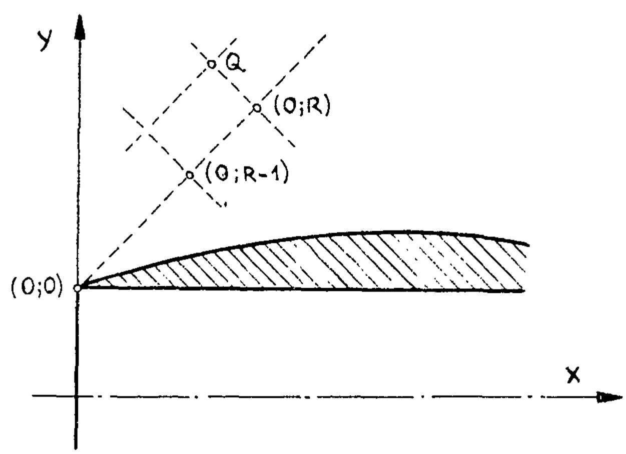


Fig. 5

On the right Mach line, starting from (0,R), let us assume a point Q Recalling that the flow is undisturbed at Q and assuming that Q is approaching (0,R) (therefore $y_Q \rightarrow y_{0;R}$), one gets immediately from the right characteristic equation:

$$(vy)_{0;R} = -\beta y_{0;R} u_{0;R} \qquad (5)$$

It is evident that this relation is valid for all points on the first Mach line,and particularly for (0,R-1), which yields:

$$(vy)_{0;R-1} = -\beta y_{0;R-1}\, u_{0;R-1} \tag{5'}$$

Substituting values (5) and (5') into the equation (3), one can calculate the value of $u_{0;R}$ as a function of $u_{0;R-1}$

$$u_{0;R} = \frac{y_{0;R} + 3y_{0;R-1}}{3y_{0;R} + y_{0;R-1}}\, u_{0;R-1} \tag{6}$$

It is possible, however, to obtain the value of $u_{0;R}$ as a function of $y_{0;R}$ only. For this purpose, let us call

$$\Delta u = u_{0;R} - u_{0;R-1}$$

$$\Delta y = y_{0;R} - y_{0;R-1}$$

The equation (6) becomes now

$$u_{0;R} = \frac{4y_{0;R} - 3\Delta y}{4y_{0;R} - \Delta y}\left(u_{0;R} - \Delta u\right) \tag{7}$$

Assuming that $\Delta y \to 0$ and neglecting the second order terms, the following differential equation is obtained (omitting the subscripts)

$$\frac{du}{u} = -\frac{1}{2}\frac{dy}{y} \tag{7'}$$

whose solution is:

$$u = \left[\frac{y}{y_0}\right]^{-\frac{1}{2}} \cdot u_0 \qquad (8)$$

where u_0 and y_0 are the values at the diffuser nose.

6. POINTS ON THE DIFFUSER SURFACE

As to points (K;K) which lie on the diffuser surface, admitting that the flow is reaching them with the right Mach line (see figure 6) equations (3') and (3'') must be

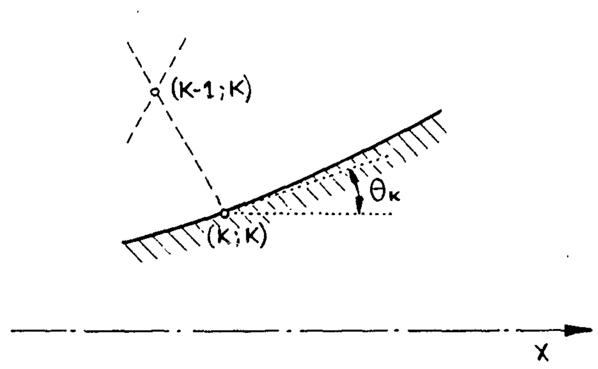


Fig. 6

used. After solving them and eliminating $(vy)_{K-1;K}$ by means of equation (3), one gets

$$u_{K;K} = A_{K;K}\, u_{K-1;K} - B_{K;K}\, u_{K-1;K-1} + C_{K;K}\, u_0 \qquad (9)$$

where A, B and C are certain geometrical expressions depend ing on positions of points (K;K), (K-1;K), (K-1;K-1) and on slopes of the diffuser surface at points (K;K) and (K-1;K-1). Their values are:

$$\left.\begin{aligned} A_{K;K} &= \frac{y_{K;K} + 2y_{K-1;K} + y_{K-1;K-1}}{\left(1+\frac{2\theta_K}{\beta}\right) y_{K;K} + y_{K-1;K}} \\ B_{K;K} &= \frac{\left(1-\frac{2\theta_{K-1}}{\beta}\right) y_{K-1;K-1} + y_{K-1;K}}{\left(1+\frac{2\theta_K}{\beta}\right) y_{K;K} + y_{K-1;K}} \\ C_{K;K} &= \frac{2\left(1+\frac{\theta_0}{\beta}\right)\left(\frac{\theta_K}{\theta_0} y_{K;K} - \frac{\theta_{K-1}}{\theta_0} y_{K-1;K-1}\right)}{\left(1+\frac{2\theta_K}{\beta}\right) y_{K;K} + y_{K-1;K}} \end{aligned}\right\} \qquad (10)$$

Naturally, assuming a Mach net such that:

$$y_{0;R} - y_{0;R-1} = y_{0;R-1} - y_{0;R-2} = \dots = \text{const} = \Delta y$$

and denoting: $y_{K;K} - y_{K-1;K-1} = (\Delta y)_K$ the values of A, B and C may be presented in a form more convenient for numerical calculations, i.e.:

$$\left.\begin{aligned} A_{K;K} &= \frac{4y_{K;K} + 2\Delta y - 3(\Delta y)_K}{2\left(1+\frac{\theta_K}{\beta}\right) y_{K;K} + \Delta y - (\Delta y)_K} \\ B_{K;K} &= \frac{2\left(1-\frac{\theta_{K-1}}{\beta}\right) y_{K;K} + \Delta y - 2\left(1-\frac{\theta_{K-1}}{\beta}\right)(\Delta y)_K}{2\left(1+\frac{\theta_K}{\beta}\right) y_{K;K} + \Delta y - (\Delta y)_K} \\ C_{K;K} &= \frac{2\left(1+\frac{\theta_0}{\beta}\right)\left[\frac{\theta_K - \theta_{K-1}}{\theta_0} y_{K;K} + \frac{\theta_{K-1}}{\theta_0}(\Delta y)_K\right]}{2\left(1+\frac{\theta_K}{\beta}\right) y_{K;K} + \Delta y - (\Delta y)_K} \end{aligned}\right\} \qquad (10')$$

As to the point (1;1), using the more accurate equation (5') instaed of (3), one gets:

$$u_{1;1} = \frac{\left(1+\frac{\theta_0}{\beta}\right)\frac{\theta_1}{\theta_0}\,y_{1;1}}{\left(1+\frac{\theta_1}{\beta}\right)y_{1;1} + \frac{\Delta y-(\Delta y)_1}{2}}\,u_{0;0} - \frac{\Delta y-(\Delta y)_1}{2\left(1+\frac{\theta_1}{\beta}\right)y_{1;1}+\Delta y-(\Delta y)_1}\,u_{0;1} \qquad (9')$$

7. THE INTERIOR POINTS OF THE NET

Solving equations (3) and (3') for

$$u_{L;R} = \frac{(vy)_{L-1;R}-(vy)_{L;R-1}+\frac{1}{2}\beta\left(y_{L;R}+y_{L;R-1}\right)u_{L;R-1}+\frac{1}{2}\beta\left(y_{L;R}+y_{L-1;R}\right)u_{L-1;R}}{\frac{1}{2}\beta\left(2y_{L;R}+y_{L;R-1}+y_{L-1;R}\right)} \qquad (11)$$

writing

$$(vy)_{L-1;R}-(vy)_{L;R-1} = \left[(vy)_{L-1;R}-(vy)_{L-1;R-1}\right] - \left[(vy)_{L;R-1}-(vy)_{L-1;R-1}\right]$$

and utilizing directly equations (3) and (3') to calculate the above expressions as a function of u, upon eliminating the values of (vy) we get finally:

$$u_{L;R} = D_{L;R}\, u_{L-1;R} - E_{L;R}\, u_{L;R-1} - F_{L;R}\, u_{L-1;R-1} \tag{11'}$$

where D, E, and F are certain geometrical expressions, depending on the Mach net assumed. Their values are:

$$\left.\begin{aligned}
D_{L;R} &= \frac{3y_{L-1;R} + y_{L-1;R-1}}{2y_{L;R} + y_{L;R-1} + y_{L-1;R}} \\
E_{L;R} &= \frac{3y_{L;R-1} + y_{L-1;R}}{2y_{L;R} - y_{L;R-1} + y_{L-1;R}} \\
F_{L;R} &= \frac{3y_{L;R-1} + 3y_{L-1;R} - 2y_{L;R}}{2y_{L;R} + y_{L;R-1} + y_{L-1;R}}
\end{aligned}\right\} \tag{12}$$

Assuming the same Mach net as in the former section, the values of D, E, F may be presented in a simpler form:

$$\left.\begin{aligned}
D_{L;R} &= \frac{4y_{L;R} + 2\Delta y - 3(\Delta y)_L}{4y_{L;R} - (\Delta y)_L} \\
E_{L;R} &= \frac{4y_{L;R} - 2\Delta y - (\Delta y)_L}{4y_{L;R} - (\Delta y)_L} \\
F_{L;R} &= \frac{4y_{L;R} - 3(\Delta y)_L}{4y_{L;R} - (\Delta y)_L}
\end{aligned}\right\} \tag{12'}$$

Calculating particularly the points of the "second" Mach line (i.e. with L = 1), in order to eliminate the values (vy), it is possible to use equations (5) and (5') -more exact- instead of (3). Thus:

$$(vy)_{0;R} - (vy)_{1;R-1} = \beta\, y_{0;R-1}\, u_{0;R-1} - \beta\, y_{0;R}\, u_{0;R} +$$

$$+ \frac{1}{2}\beta (y_{1;R-1} + y_{0;R-1})(u_{1;R-1} - u_{0;R-1})$$

Substituting this expression into equation (11), and solving it for $u_{1;R}$, one gets:

$$u_{1;R} = E_{1;R}\, u_{1;R-1} + G_{1;R} \cdot (u_{0;R-1} - u_{0;R}) \qquad (11")$$

where the value of $E_{1;R}$ is determined by formula (12') and

$$G_{1;R} = \frac{y_{0;R} - y_{1;R}}{2y_{1;R} + y_{1;R-1} + y_{0;R}} = \frac{\Delta y - (\Delta y)_1}{4y_{1;R} - (\Delta y)_1} \qquad (12")$$

As to expression $(u_{0;R-1} - u_{0;R})$ in equation (11"), it is possible to present it in the form -see equation (7)-:

$$u_{0;R-1} - u_{0;R} = \frac{2\Delta y}{4y_{0;R} - \Delta y}\, u_{0;R-1}$$

The equations (4) and (8) enable to calculate the u-velocity directly from the geometrical positions of points, and the equations (9), (9'), (11'), (11") - from the known data at former points of the Mach net.

8. A PRACTICAL PROCEDURE TO CALCULATE THE VELOCITY DISTRIBUTION ALONG THE SURFACE OF A DIFFUSER

The practical procedure may be presented as follows:

On the first Mach line of a diffuser one chooses certain number of equidistant points (including the nose point) so that the Mach net, starting from these (see figure 3) covers the entire region where the velocity distribution is to be calculated.

After finding from equations (10') the coeficients $\underline{A}$, $\underline{B}$ and $\underline{C}$ for all surface points, and $\underline{D}$, $\underline{E}$, $\underline{F}$ from equations (12') for all interior points, one finds the velocities $\underline{u}$ at different net points, in the following order:

(i) - nose point - equation (4)

(ii) - points on the first Mach line - equation (8)

(iii) - point (1;1) next to the nose on diffuser surface - equation (9')

(iv) - points on the Mach line L = 1, issuing downstream from point (1;1) - equation (11")

The procedure described in items (iii) and (iv) is next repeated in the same order with respect to other surface points - equation (9) - and lines L = const - equation (11') - issuing therefrom, till the last net point in the diffuser surface is reached.

It is necessary to mention here that in order to obtain the velocities $\underline{u}$ for $\underline{n}$ points on the diffuser surface, they must be determined also for $\frac{1}{2}n(n-1)$ remaining net points. So, on the one hand, the more dense is the Mach net, the more accurate the results will be, but, on the other hand, the amount of work required increases considerably.

Part two - STRAIGHT LINE DIFFUSERS

9. ADAPTATION OF THE METHOD TO STRAIGHT LINE DIFFUSERS.

In case of a straight line diffuser the amount of work may be reduced, assuming that the distance between two neighboring net points on the surface of the diffuser is constant (see figure 7). Thus, the x co-ordinates of these points will be K.m, where K = 0;1;2; ... and m is certain arbitrary length on which depends the net density and, therefore, the accuracy of the solution.

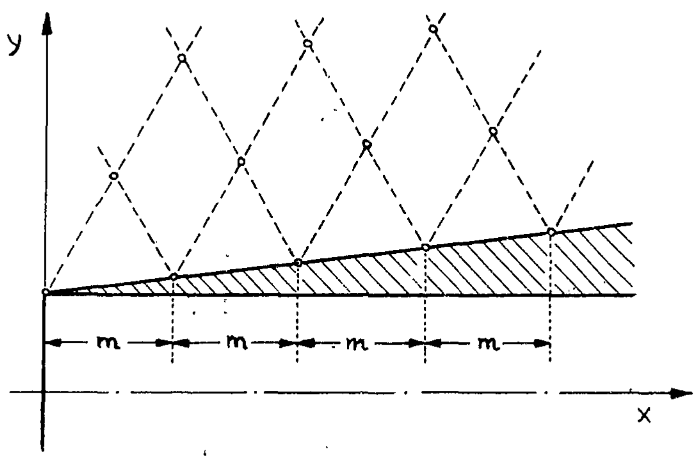


Fig. 7

It results from geometrical relations, that (→ section 4):

$$y_{L;R} = y_0 - \frac{m}{2\beta}(1-\beta\theta)L + \frac{m}{2\beta}(1+\beta\theta)R$$

$$(\Delta y)_K = \text{const} = m\tan\theta \simeq m\theta$$

$$\Delta y = \frac{m}{2\beta}(1+\beta\theta)$$

Substituting these values into equations (8), (9), (9'), (11'), and (11"), and dividing them by u_0 one gets:

$\boxed{A}$ For the first Mach line:

$$\frac{u_{0;R}}{u_0} = \left[\frac{y}{y_0}\right]^{-\frac{1}{2}} \tag{13}$$

$\boxed{B}$ For the diffuser surface:

$$\frac{u_{K;K}}{u_0} = \frac{2y_{K;K} + \frac{1-2\beta\theta}{2\beta}m}{\left(1+\frac{\theta}{\beta}\right)y_{K;K} + \frac{1-\beta\theta}{4\beta}m}\,\frac{u_{K-1;K}}{u_0} - \frac{\left(1-\frac{\theta}{\beta}\right)y_{K;K} + \frac{1-3\beta\theta+4\theta^2}{4\beta}m}{\left(1+\frac{\theta}{\beta}\right)y_{K;K} + \frac{1-\beta\theta}{4\beta}m}\,\frac{u_{K-1;K-1}}{u_0} +$$

$$+ \frac{\left(1+\frac{\theta}{\beta}\right)m\theta}{\left(1+\frac{\theta}{\beta}\right)y_{K;K} + \frac{1-\beta\theta}{4\beta}m} \tag{14}$$

particularly, for K=1, one may use the more exact equation:

$$\frac{u_{1;1}}{u_0} = \frac{\left(1+\frac{\theta}{\beta}\right)y_{1;1}}{\left(1+\frac{\theta}{\beta}\right)y_{1;1} + \frac{1-\beta\theta}{4\beta}m} - \frac{\frac{1-\beta\theta}{4\beta}m}{\left(1+\frac{\theta}{\beta}\right)y_{1;1} + \frac{1-\beta\theta}{4\beta}m}\,\frac{u_{0;1}}{u_0} \tag{14'}$$

$\boxed{C}$ For the interior points:

$$\frac{u_{L;R}}{u_0} = \frac{y_{L;R} - \frac{1-2\beta\theta}{4\beta} m}{y_{L;R} - \frac{m\theta}{4}} \frac{u_{L-1;R}}{u_0} + \frac{y_{L;R} - \frac{1+2\beta\theta}{4\beta} m}{y_{L;R} - \frac{m\theta}{4}} \frac{u_{L;R-1}}{u_0} - \frac{y_{L;R} - \frac{3}{4} m\theta}{y_{L;R} - \frac{1}{4} m\theta} \frac{u_{L-1;R-1}}{u_0} \tag{15}$$

and particularly for L = 1, one may use more exact equation:

$$\frac{u_{1;R}}{u_0} = \frac{y_{1;R} - \frac{1+2\beta\theta}{4\beta} m}{y_{1;R} - \frac{m\theta}{4}} \frac{u_{1;R-1}}{u_0} + \frac{\frac{1-\beta\theta}{8\beta} m}{y_{1;R} - \frac{m\theta}{4}} \left(\frac{u_{0;R-1}}{u_0} - \frac{u_{0;R}}{u_0}\right) \tag{15'}$$

where the coefficients of velocities (u/u_0), in spite of their apparent complication, can be determined numerically for a given diffuser and a given flow quite easily and rapidly.

Three graphs are enclosed - 1 to 3 - (see pg.60/62), which show the u-velocity distribution, found by means of the above described method. The angles θ are: 0,025 radians, 0,05 radians and 0,125 radians; β is assumed equal to 1, and the segment m varies from $0{,}2\,y_0$ for initial points to $0{,}8\,y_0$ for last points of net. The computations were performed in the region $0 \leq x \leq 16\,y_0$. The dependent variable is u/u_0

10. AN OUTLINE OF THE ANALYTICAL SOLUTION IN THE CASE OF A STRAIGHT LINE DIFFUSER.

The method described in the preceding section furnishes a possibility of finding in a simple way numerical results for the velocity distribution along the surface of a straight line diffuser. It doesn't, however, determine the analytical form of this solution, nor its relation with the angle θ.

In order to answer this question, let us assume, for the sake of argument, that the diffuser extends downstream to infinity. Obviously, in the case of supersonic flow, this does not restrict the generality of a solution.

Let us call $u(x)$ a function of $\underline{x}$ which represents the u-velocity distribution under investigation along the surface of the diffuser. The value of this function at the nose point and at infinity can be immediately established.

In fact, at the nose point, as it was already shown in the preceding section, the velocity $u_{(o)}$ (which will be, as earlier, indicated by u_o) is equal to the corresponding velocity in a two dimensional deflection, that is:

$$u_o \simeq -\frac{\theta}{\beta} U_\infty$$

As to the velocity $u_{(\infty)}$ (which will be indicated as u_∞ , it is possible to use a more intuitive similarity or "scale effect" reasoning. Consider at first that the length of the diffuser is limited and equal to $\underline{a}$ (see figure 8). The radius of its nose point is y_0. If we now extend $\underline{a}$ to infinity, but choose the scale of the drawing such that this length remains always the same, then y_0 (in the figure) will tend to zero, becoming zero when $a = \infty$. In this case we obtain a cone. Thus the velocity on the surface at infinity for a given supersonic flow is uniquely defined, since the velocity of the surface of a cone does not depend on the position of the point but only on the angle θ.

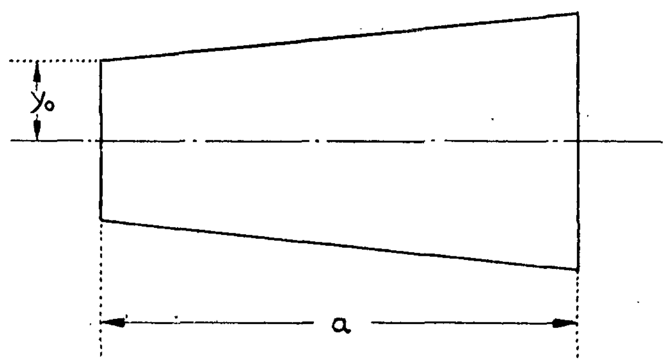


Fig. 8

One can obtain this velocity from appropriate plots, or from cone tables, or approximately (for small values of θ):

$$u_\infty = -\theta^2 \ln \frac{2}{\beta\theta}$$

In order to establish the form of the function $u(x)$ for values $0 < x < \infty$, it is convenient to assume an auxiliary function:

$$\sigma(x,\theta,\beta) = \frac{u(x) - u_\infty}{u_0 - u_\infty} \tag{16}$$

which, independently of θ and β , for x = o and x = ∞ admits the following values:

$$\sigma(0,\theta,\beta) = \frac{u_0 - u_\infty}{u_0 - u_\infty} = 1$$

$$\sigma(\infty,\theta,\beta) = \frac{u_\infty - u_\infty}{u_0 - u_\infty} = 0$$

The behavior of the function σ for other values of x is - for the time being - unknown. It will be investigated, however, in the next sections. Once the function $\sigma(x,\theta,\beta)$ is established, finding of the function $u(x)$ becomes trivial, because from equation (16):

$$u(x) = u_\infty + (u_0 - u_\infty)\cdot\sigma(x,\theta,\beta) \tag{17}$$

We see, from equation (17), that the function σ has a very simple physical interpretation: it indicates the manner in which the transition from two dimensional flow at the nose point ($\sigma = 1 \therefore u_{(x)} = u_0$) to three dimensional conical flow at infinity ($\sigma = 0 \therefore u_{(x)} = u_\infty$) is performed along the surface of straight line diffusers.

11. DETERMINATION OF THE LIMIT OF THE FUNCTION σ AS THE ANGLE θ APPROACHES ZERO.

When $\theta = 0$, the flow is undistrubed and the function σ has, in this case, an undetermined value:

$$\sigma(x, 0, \beta) = \frac{0}{0}$$

Regardless of this, as will be proved, the limit of this expression exists. Let us indicate it:

$$\overset{+}{\sigma}(x, \beta) = \lim_{\theta \to 0} \sigma(x, \theta, \beta)$$

To prove its existence, let us note, that

$$\lim_{\theta \to 0} \sigma(x, \theta, \beta) = \lim_{\theta \to 0} \frac{u_{(x)} - u_\infty}{u_0 - u_\infty} =$$

$$= \frac{\lim\limits_{\theta \to 0} \frac{u_{(x)}}{u_0} - \lim\limits_{\theta \to 0} \frac{u_\infty}{u_0}}{1 - \lim\limits_{\theta \to 0} \frac{u_\infty}{u_0}}$$

Using L'Hospital rule, we have:

$$\lim_{\theta\to 0} \frac{u_\infty}{u_0} = \lim_{\theta\to 0} \frac{\ln \frac{2}{\beta\theta}}{\frac{1}{\beta\theta}} = \lim_{\theta\to 0} (\beta\theta) = 0$$

from which

$$\lim_{\theta\to 0} \sigma(x,0,\beta) = \lim_{\theta\to 0} \frac{u(x)}{u_0} \tag{18}$$

The function $u(x)/u_0$ can be calculated numerically by means of equations (13), (14), (14'), (15) and (15'). We recall that all these equations are determined when $\theta\to 0$ and have the following limits when $\theta = 0$:

$\boxed{A}$ For L = 0:

$$\left[\frac{u_{0;R}}{u_0}\right] = \left[1 + \frac{Rm^*}{2}\right]^{-\frac{1}{2}} \tag{19}$$

$\boxed{B}$ For L = R $(= K \neq 0)$:

$$\left[\frac{u_{K;K}}{u_0}\right] = 2\left[\frac{u_{K-1;K}}{u_0}\right] - \left[\frac{u_{K-1;K-1}}{u_0}\right] \tag{20}$$

and particularly for K = 1

$$\left[\frac{u_{1;1}}{u_0}\right] = \frac{1 - \frac{m^*}{4}\left[\frac{u_{0;1}}{u_0}\right]}{1 + \frac{m^*}{4}} \tag{20'}$$

$\boxed{C}$ For $L \neq R$; $L \neq 0$

$$\left[\frac{u_{L;R}}{u_0}\right] = \frac{2+(R-L+\frac{1}{2})m^*}{2+(R-L)m^*}\left[\frac{u_{L-1;R}}{u_0}\right] - \frac{2+(R-L-\frac{1}{2})m^*}{2+(R-L)m^*}\left[\frac{u_{L;R-1}}{u_0}\right] - \left[\frac{u_{L-1;R-1}}{u_0}\right] \quad (21)$$

and particularly for L = 1:

$$\left[\frac{u_{1;R}}{u_0}\right] = \frac{2+(R-\frac{3}{2})m^*}{2+(R-1)m^*}\left[\frac{u_{1;R-1}}{u_0}\right] + \frac{\frac{1}{4}m^*}{2+(R-1)m^*}\left\{\left[\frac{u_{0;R-1}}{u_0}\right] - \left[\frac{u_{0;R}}{u_0}\right]\right\} \quad (21')$$

where:

$$m^* = \frac{m}{\beta y_0}$$

Since all these equations are determined it is evident that the limit (18) exists and may be numerically calculated at points Km by solving for $u_{K;K}/u_0$. Moreover, it is clear, from the structure of the above equations that this solution is a function of a parameter $m^* = m/\beta y_0$. Thus, introducing a new variable:

$$\xi = \frac{x}{\beta y_0}$$

to each point x = Km on the x axis, there will correspond a point:

$$\xi = K\frac{m}{\beta y_0} = K m^*$$

on the ξ axis. The values $u_{K;K}/u_0$ will be now subordinated to the points Km^* but as they depend only on the choice of m^* the function σ can be expressed as a function of only one variable ξ. Thus:

$$\lim_{\theta \to 0} \sigma(x,\theta,\beta) = f(\xi) = f(x/\beta y_0)$$

12. NUMERICAL EVALUATION OF THE FUNCTION $\sigma(\xi)$

Substituting into equations (19-21) any numerical value for m^* and solving for $u_{K,K}/u_0$ we shall obtain the values of the function $\sigma(\xi)$ at the points $\xi = K m^*$. The following tables give the values of σ as a function of ξ for $m^* = 0{,}4$ and for $m^* = 0{,}2$ in the range $0 \le \xi \le 4$

ξ	$\sigma(\xi)$	
	$m^* = 0{,}2$	$m^* = 0{,}4$
0	1	1
0,2	0.907	-
0,4	0.826	0.825
0,6	0.756	-
0,8	0.694	0.693
1,0	0.640	-
1,2	0.592	0.592
1,4	0.549	-
1,6	0.511	0.511
1,8	0.477	-
2	0.447	0.447

ξ	$\sigma(\xi)$	
	$m^* = 0{,}2$	$m^* = 0{,}4$
2,2	0.419	-
2,4	0.394	0.394
2,6	0.371	-
2,8	0.351	0.351
3,0	0.332	-
3,2	0.315	0.315
3,4	0.299	-
3,6	0.285	0.285
3,8	0.271	-
4	0.259	0.259

Substituting for m* values less than 0,2 one will not obtain any changes in the first three decimal points of the function $\sigma(\xi)$. The result obtained for $m^* = 0,2$ can be therefore considered as a final one for the first three decimal places.

For greater values of ξ, ($\xi > 4$), it is necessary to increase the value of m* . Although such results for $u_{K;K}$ at small values of K are always determined with relatively greater error, as K increases a tendency towards "self correction" is observed. Thus, combining different results (with different values of m*), it is possible to reproduce relatively exact values of $\sigma(\xi)$.

These values are:

ξ	$\sigma(\xi)$
5	0.208
6	0.175
8	0.131

ξ	$\sigma(\xi)$
10	0.104
20	0.051
40	0.025

From the last results a tendency can be distinctly seen for the function $\sigma(\xi)$ to approach ξ^{-1} for great values of ξ.

Graph 4 shows the shape of the function $\sigma(\xi)$ in the region $0 \leq \xi \leq 16$.

13. THE SUCCESSIVE ANALYTICAL APPROXIMATIONS OF THE FUNCTION $\sigma(\xi)$.

Let us solve the system of equations (19-21') for $u_{K;K}$ $(K=1;2;3;\ldots)$, without substituting any numerical value for m^*. These solutions will be, thus, functions of m^*; let us indicate them $T_1(m^*); T_2(m^*); \ldots$ .Generally, each function $T_K(m^*)$ determines approximately (with accuracy increasing with K) the value of the function $\sigma(\xi)$ at point $\xi = K m^*$. Thus, for each concrete value of K we have:

$$\sigma(Km^*) = T_K(m^*)$$

But, as the value Km^* (for concrete K) is not fixed and may admit any arbitrary value ξ, we have:

$$\sigma(\xi) \simeq T_K(\xi/K)$$

So, the sequence of the functions: $T_1(\xi)$; $T_2(\xi/2)$; $T_3(\xi/3)$; ... represents the successive, more and more accurate approximations of the function $\sigma(\xi)$

In the case of $K=1$ one obtains:

$$T_1(\xi) \doteq \frac{-\frac{1}{4}\xi + \sqrt{1+\frac{1}{2}\xi}}{(1+\frac{1}{4}\xi)\sqrt{1+\frac{1}{2}\xi}}$$

Expanding the value of $\sqrt{1+\frac{1}{2}\xi}$ into power series and keeping only the first two terms, one obtains immediately:

$$\sigma(\xi) \simeq J_1(\xi) \simeq \frac{1}{\left(1+\frac{1}{4}\xi\right)^2} \tag{22}$$

as a first approximate of the function $\sigma(\xi)$

Choosing $K=1$, the functions J become at once very complicated. For instance, already for $K=2$, there is:

$$\sigma(\xi) \simeq \frac{\xi^3 - 4\xi^2 + 32\xi + 256}{256\left(1+\frac{1}{8}\xi\right)^2\left(1+\frac{1}{4}\xi\right)^2} \tag{23}$$

For practical purposes, the approximation given by equation (22) is quite adequate. Being very simple, it provides at the same time quite good accuracy, especially in the range $0 \leq \xi \leq 4$; this may be demonstrated by a comparison with the values of the function $\sigma(\xi)$ given in section 12.

ξ	$\sigma(\xi)$	$(1+\frac{1}{4}\xi)^{-2}$
0	1	1
0,2	0.907	0.907
0,4	0.826	0.826
0,6	0.756	0.756
0,8	0.694	0.694
1	0.640	0.640
1,2	0.592	0.592
1,4	0.549	0.549
1,6	0.511	0.510
1,8	0.477	0.476
2	0.447	0.444

ξ	$\sigma(\xi)$	$(1+\frac{1}{4}\xi)^{-2}$
2,2	0.419	0.416
2,4	0.394	0.391
2,6	0.371	0.367
2,8	0.351	0.346
3	0.332	0.327
3,2	0.315	0.309
3,4	0.299	0.292
3,6	0.285	0.277
3,8	0.271	0.263
4	0.259	0.250

The behavior of the function $\sigma(\xi)$ for large values of ξ may be investigated by means of its higher approximations: $\sigma_2(\xi/2)$; $\sigma_3(\xi/3)$; .... ; it can be shown that they all approach, (for large values of ξ), the functions $a\xi^{-1}$ where $\underline{a}$ varies, depending on $\underline{K}$; as to the function $\sigma(\xi)$ itself, it seems from numerical values given in section 12, that $a = 1$, i.e. that $\sigma(\xi)$ approaches, for large values of ξ the function ξ^{-1} (this will be confirmed in next sections).

The comparison between the function $\sigma(\xi)$ and its first approximation $[1+\frac{1}{4}\xi]^{-2}$ for large values of ξ , shows that both functions approach zero; however, the former does it like the function ξ^{-1} while the latter as 16 ξ^{-2} . Thus, the absolute error:

$$\left| \sigma(\xi) - \left(1+\frac{1}{4}\xi\right)^{-2} \right|$$

which, as can be established, is always less then 0.025, will also approach zero for large values of ξ . The relative error will, of course, tend to infinity. For practical computations of the u-velocity, however, with certain restriction, it will have no major effect (see section 17).

Graph 5 shows the comparison between the functions $\sigma(\xi)$, $(1+\frac{1}{4}\xi)^{-2}$ and ξ^{-1} in the range $0 \leq \xi \leq 16$

14. THE OPERATIONAL FORM OF THE FUNCTION $\sigma(\xi)$

It is possible to obtain the exact analytical representation of the function $\sigma(\xi)$ by means of operational calculus.

For this purpose one will use the transformation:

$$F(p) = p \int_0^\infty f(x)\, e^{-px}\, dx \qquad (24)$$

which will be denoted by the symbols:

$$F(p) \doteq \mathfrak{P}[f(x)]$$

The inverse transformation will be denoted as:

$$f(x) = \mathfrak{P}^{-1}\left[F(p)\right]$$

As it is well known, the inverse transformation may be determined, under certain restrictive conditions, by means of the formula:

$$\mathfrak{P}^{-1}\left[F(p)\right] = \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} F(z)\,\frac{e^{xz}}{z}\,dz \tag{25}$$

where x_0 must be chosen so that the line $Re(z) = x_0$ passes to the right of all singularities of the integrand.

The function $\sigma(\xi)$ may be determined by solving the differential equation for the perturbation potential φ in three dimensional axially symmetrical flow:

$$\frac{\partial^2\varphi}{\partial y^2} + \frac{1}{y}\frac{\partial\varphi}{\partial y} - \beta^2\frac{\partial^2\varphi}{\partial x^2} = 0 \tag{26}$$

with boundary conditions

$$\left.\begin{aligned} (v)_{y=\infty} &= \left(\frac{\partial\varphi}{\partial y}\right)_{y=\infty} = 0 \\ (v)_{y=y_0+\theta x} &= \left(\frac{\partial\varphi}{\partial y}\right)_{y=y_0+\theta x} = \theta\cdot U_\infty \end{aligned}\right\} \tag{27}$$

(once it is assumed $\theta \rightarrow 0$, the second term of the sum $y_0+\theta x$ will be omitted).

Transforming equation (26) by means of equation (24) with respect to the variable $\underline{x}$, and taking into consideration that:

$$\mathcal{P}\left[\frac{\partial^2\varphi}{\partial y^2}\right] = \frac{d^2}{dy^2}\mathcal{P}[\varphi]$$

$$\mathcal{P}\left[\frac{1}{y}\frac{\partial\varphi}{\partial y}\right] = \frac{1}{y}\frac{d}{dy}\mathcal{P}[\varphi]$$

$$\mathcal{P}\left[\beta^2\frac{\partial^2\varphi}{\partial x^2}\right] = \beta^2\left\{p^2\mathcal{P}[\varphi] - p^2\varphi_{x=+0} - p\left(\frac{\partial\varphi}{\partial x}\right)_{x=+0}\right\}$$

and that the choice of the co-ordinate system is such that for $x = +0$:

$$\varphi = 0$$

$$\frac{\partial\varphi}{\partial x} = u = 0$$

one obtains the equation:

$$\frac{d^2}{dy^2}\mathcal{P}[\varphi] + \frac{1}{y}\frac{d}{dy}\mathcal{P}[\varphi] - \beta^2 p^2\mathcal{P}[\varphi] = 0 \tag{28}$$

the solution of which is:

$$\mathcal{P}[\varphi] = C_1 I_0(\beta y p) + C_2 K_0(\beta y p) \tag{29}$$

Submitting the boundary conditions to the transformation $\underline{P}$, one obtains:

$$\left\{\frac{\partial}{\partial y} P[\varphi]\right\}_{y=\infty} = 0$$

$$\left\{\frac{\partial}{\partial y} P[\varphi]\right\}_{y=y_0} = P[\theta U_\infty] = \theta U_\infty$$

Applying the above conditions and taking into consid eration that

$$\frac{\partial}{\partial y} P[\varphi] = C_1 \beta p I_1(\beta y p) - C_2 \beta p K_1(\beta y p)$$

one gets:

$$C_1 = 0$$

$$C_2 = -\frac{\theta U_\infty}{\beta p K_1(\beta y_0 p)}$$

and finally:

$$P[\varphi] = -\frac{K_0(\beta y p)}{\beta p K_1(\beta y_0 p)} \theta U_\infty \tag{30}$$

The operational representation of the velocity $u_{(x,y)}$ with respect to $\underline{x}$, is:

$$\mathcal{P}\left[u(x,y)\right] = \mathcal{P}\left[\frac{\partial \varphi}{\partial x}\right] = p\ \mathcal{P}\left[\varphi\right] =$$

$$= -\frac{K_0(\beta y p)}{K_1(\beta y_0 p)}\ \frac{\theta U_\infty}{\beta} = \frac{K_0(\beta y p)}{K_1(\beta y_0 p)}\ u_0 \qquad (31)$$

The last result enables us immediately to establish the operational form of the function $\sigma(\xi)$. In fact, combining equations (18) and (31), one gets:

$$\mathcal{P}\left[\sigma(x,\beta)\right] = \mathcal{P}\left[\frac{u(x,y_0)}{u_0}\right] = \frac{K_0(\beta y_0 p)}{K_1(\beta y_0 p)} \qquad (32)$$

i.e., the final result.

15. CALCULATION OF THE FUNCTION $\sigma(x,\beta)$ FROM ITS OPERATIONAL FORM.

In order to find the function $\sigma(x,\beta)$ one has to perform the operation:

$$\mathcal{P}^{-1}\left[\frac{K_0(\beta y_0 p)}{K_1(\beta y_0 p)}\right] \qquad (33)$$

With this purpose the operation will be performed first

$$p^{-1}\left[\frac{K_0(p)}{K_1(p)}\right] \tag{33'}$$

i.e., the integral will be calculated

$$\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}\frac{K_0(z)}{K_1(z)}\,\frac{e^{xz}}{z}\,dz \tag{33''}$$

and next, "shift" formula of operational calculus will be applied.

As it is known, the functions $K_0(z)$ and $K_1(z)$ have a logarithmic branch point at $z = 0$; besides this they have no other singularities. Moreover, the function $K_1(z)$ does not have any zeros in the region $|\arg z|<\pi$ so the integrand of (33") is holomorphic therein.

As a result, we may write:

$$\int_{\mathcal{K}}\frac{K_0(z)}{K_1(z)}\,\frac{e^{xz}}{z}\,dz = 0 \tag{34}$$

where $\mathcal{K}$ = ABCDEFGHA is the path of integration, shown in figure 9.

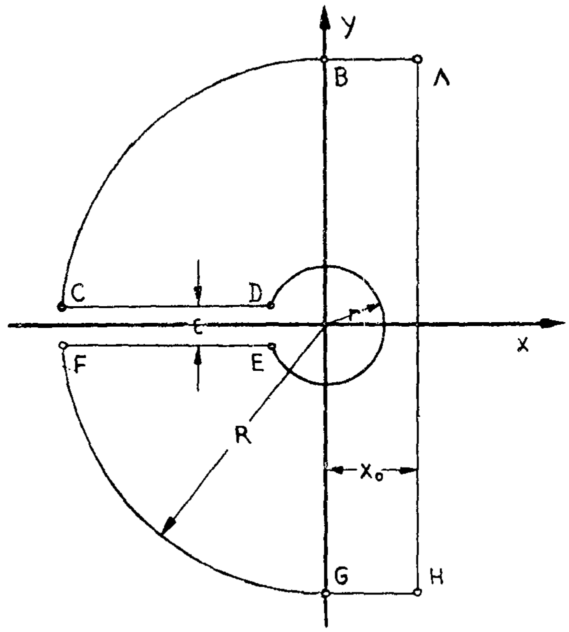


Fig. 9

If it is now assumed, that:

$$\begin{cases} x_o \rightarrow 0 \\ r \rightarrow 0 \\ \varepsilon \rightarrow 0 \\ R \rightarrow \infty \end{cases}$$

then the path $\mathcal{K}$ may be divided (in the limit) in the following manner:

$$\mathcal{K} = \mathcal{L} + \mathcal{C}' + \mathcal{C}'' + \mathcal{M} + \mathcal{N}$$

where:

$\mathcal{L}$ - is the path on the y axis from $-\infty$ to $+\infty$ leaving the point $z = 0$ to the left by means of the semicircle with $r \to 0$

C' - is the semicircle of radius $R \to \infty$ traced from the point $z = 0$ to the left of the $\underline{y}$ - axis

C'' - is a circle of the radius $r \to 0$ traced from the point $z = 0$

$\mathcal{M}$ - is the path from $-\infty$ to 0 on the $\underline{x}$ axis, considering that: $-x = re^{i\pi}$

$\mathcal{N}$ - is the path from 0 to $-\infty$ on the $\underline{x}$ axis, considering that: $-x = re^{-i\pi}$

The behavior of the integral (33) along all these paths will now be investigated.

a) We have:

$$\int_{C'} \frac{K_0(z)}{K_1(z)} \frac{e^{xz}}{z} dz = 0 \tag{35}$$

which may be proved by means of Jordan's theorem, taking into consideration that:

$$\lim_{z \to \infty} \left| \frac{K_0(z)}{z K_1(z)} \right| = \lim_{z \to \infty} \left| \frac{\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}}{z \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}} \right| = 0$$

b) The integral

$$\int_{C''} \frac{K_0(z)}{K_1(z)} \frac{e^{xz}}{z} \, dz = 0 \tag{36}$$

because

$$\left| \int_{C''} \frac{K_0(z)}{K_1(z)} \frac{e^{xz}}{z} \, dz \right| \leq 2\pi e^{rx} \lim_{z \to 0} \left| \frac{K_0(z)}{K_1(z)} \right| =$$

$$= 2\pi e^{rx} \lim_{z \to 0} \left| \frac{-\ln \frac{z}{2} - \frac{1}{2}\gamma}{\frac{1}{z} + \frac{z}{2} \ln \frac{z}{2}} \right| = 0$$

c) In order to calculate the integrals

$$J = \int_{M} \frac{K_0(z)}{K_1(z)} \frac{e^{xz}}{z} \, dz + \int_{N} \frac{K_0(z)}{K_1(z)} \frac{e^{xz}}{z} \, dz$$

one uses the following relations:

$$K_0(z e^{\pm \pi i}) = K_0(z) \mp i\pi I_0(z)$$

$$K_1(z e^{\pm \pi i}) = -K_1(z) \mp i\pi I_1(z)$$

therefore, after a simple transformation:

$$J = -2\pi i \int_0^\infty \frac{K_0(t)\,I_1(t) + K_1(t)\,I_0(t)}{K_1^2(t) + \pi^2 I_1^2(t)} \; \frac{e^{-tx}}{t}\, dt$$

Taking into consideration that

$$K_0(t)\,I_1(t) + K_1(t)\,I_0(t) = \frac{1}{t}$$

it gives:

$$J = -2\pi i \int_0^\infty \frac{e^{-tx}}{K_1^2(t) + \pi^2 I_1^2(t)} \; \frac{dt}{t^2} \tag{37}$$

d) Finally, as

$$\int_{\mathcal{L}} \frac{K_0(z)}{K_1(z)} \; \frac{e^{xz}}{z}\, dz = \frac{1}{2\pi i}\, \mathcal{P}^{-1}\left[\frac{K_0(p)}{K_1(p)}\right] \tag{38}$$

considering equations (34 ÷ 38), one gets:

$$\mathcal{P}^{-1}\left[\frac{K_0(p)}{K_1(p)}\right] = \int_0^\infty \frac{e^{-tx}}{K_1^2(t) + \pi^2 I_1^2(t)} \; \frac{dt}{t^2}$$

After applying the scale effect of operational calculus, the final result is:

$$\sigma(x,\beta) = p^{-1}\left[\frac{K_0(\beta y_0 p)}{K_1(\beta y_0 p)}\right] = \int_0^\infty \frac{e^{-\frac{tx}{\beta y_0}}}{K_1^2(t)+\pi^2 I_1^2(t)}\,\frac{dt}{t^2} \qquad (39)$$

It is interesting to mention here that for $y_0 = 0$ the value of $e^{-tx/\beta y_0}$ is 0, and, consequently, one obtains $\sigma(x,\beta) = 0$, and therefore $u(x) = u_\infty + \sigma(x,\beta)(u_0 + u_\infty) = u_\infty$; this esult is obvious, taking into considerations that for $y_0 = 0$ the perturbation body is a cone, and that u_∞ is (→ section 10) the u-velocity at a cone surface.

For the same reason, when the curvature is infinite, one has two dimensional flow. Therefore, substituting into equation (39) the value $y_0 = \infty$, one must obtain:

$$\int_0^\infty \frac{1}{K_1^2(t)+\pi^2 I_1^2(t)}\,\frac{dt}{t^2} = 1$$

which may be interesting even from the purely mathematical viewpoint.

16. THE APPROXIMATE SOLUTION FOR $\sigma(x,\beta)$ FROM ITS OPERATIONAL FORM.

Considering the asymptotic expansions of the functions K_0 and K_1, one obtains for large values of $\underline{z}$:

$$\frac{K_0(z)}{K_1(z)} \simeq 1 - \frac{1}{2} z^{-1} + \frac{3}{8} z^{-2} - \frac{3}{8} z^{-3} + \frac{63}{128} z^{-4} - \dots$$

Remembering that

$$\mathcal{P}^{-1}\left[a p^{-n}\right] = \frac{a x^n}{n!}$$

the expansion of the function $\sigma(x,\beta)$ for small values of $\underline{x}$ (after considering the effect of scale) is:

$$\sigma(x,\beta) = 1 - \frac{1}{2}\frac{x}{\beta y_0} + \frac{3}{16}\left[\frac{x}{\beta y_0}\right]^2 - \frac{1}{16}\left[\frac{x}{\beta y_0}\right]^3 + \frac{21}{1024}\left[\frac{x}{\beta y_0}\right]^4 - \dots \quad (40)$$

It is now possible to compare the present result with the approximate solution given in section 13, formula (22). Expanding this expression into power series, it results in:

$$\frac{1}{\left(1+\frac{x}{4\beta y_0}\right)^2} = 1 - \frac{1}{2}\frac{x}{\beta y_0} + \frac{3}{16}\left[\frac{x}{\beta y_0}\right]^2 - \frac{1}{16}\left[\frac{x}{\beta y_0}\right]^3 + \frac{20}{1024}\left[\frac{x}{\beta y_0}\right]^4 - \dots \quad (40')$$

Both series (40) and (40') diverge only from the fifth term.

Calculating now $\mathcal{P}^{-1}\left[\frac{K_0(\beta y_0 p)}{K_1(\beta y_0 p)}\right]$ for small values of $\underline{p}$ one obtains the asymptotic approximation of the function $\sigma(x,\beta)$ for large $\underline{x}$. The result is:

$$\sigma_{(x,\beta)} = \frac{\beta y_0}{x} \tag{41}$$

or:

$$\sigma(\xi) = \frac{1}{\xi} \tag{41'}$$

which was already supposed in section 13.

17. THE APPROXIMATE ANALYTICAL SOLUTION FOR THE FUNCTION $u_{(x)}$.

As known from section 10, equation (17), the function $u_{(x)}$, which represents the distribution of the u-velocity along the surface of a straight line diffuser, may be presented in the form:

$$u_{(x)} = u_{\infty} + \sigma_{(x,\theta,\beta)} \cdot (u_0 - u_{\infty})$$

where the symbol u_0 denotes the perturbation velocity at the nose point of a diffuser (equal to the two dimensional perturbation velocity deflected through the angle θ) and u_{∞} - the corresponding conical perturbation velocity.

For small values of angle θ, remembering that:

$$\sigma_{(x,\beta)} = \lim_{\theta \to 0} \sigma_{(x,\theta,\beta)}$$

it is possible to replace $\sigma(x,\theta,\beta)$ by the function $\sigma(x,\beta)$ In this case, as a first approximation of the function $u(x)$ one obtains:

$$u(x) \simeq u_\infty + \sigma(x,\beta)\cdot(u_0 - u_\infty) \qquad (42)$$

Though, as it was mentioned, this formula is valid first of all for the values $\theta \to 0$, it may be used in a relatively wide range of angles. This is due to the fact that the value of $(u_0 - u_\infty)$ in relation to u_∞ decreases with increasing values of θ .

For instance, for $\theta = 10°$ (and $\beta = 1$):

$$\frac{u_0 - u_\infty}{u_\infty} = 1,4$$

while for $\theta = 20°$ ($\beta = 1$):

$$\frac{u_0 - u_\infty}{u_\infty} = 0,6$$

Thus, when the angle θ increases, the errors due to replacement of $\sigma(x,\theta,\beta)$ by $\sigma(x,\beta)$ are compensated, be cause (also in view of the fact that $\sigma(x,\beta) \leq 1$) the sec ond term on the right hand side of equation (44) - where the errors are generated - is relatively much smaller than the first one.

Finally, replacing $\sigma(x,\beta)$ by its approximation

$\left[1+\frac{1}{4}\xi\right]^{-2}$ - in the range of $0 \leq \xi \leq 4$ the error due to this is absolutely neglegible. In the range $\xi > 4$ this error increases, but since:

$$\left| \sigma_{(\xi)} - \frac{1}{(1+\frac{1}{4}\xi)^2} \right| < 0{,}025$$

(see section 13), it may influence the numerical calculations only if:

(i) the diffuser is very long, and

(ii) the expression $\frac{u_0 - u_\infty}{u_\infty}$ is very great (which,as was said in this section, takes place only if the angle θ is extremely small.

In conclusion, remembering that $\xi = \frac{x}{\beta y_0}$, and under the condition that (i) and (ii) does not hold, the first approximation for the distribution of the u velocity along a straight line diffuser is:

$$u_{(x)} = u_\infty + \frac{u_0 - u_\infty}{\left(1 + \frac{x}{4\beta y_0}\right)^2} \tag{43}$$

In the case in which θ is very small (for instance when $\theta < 0.025$ radians) and at the same time ξ is very large, we can represent $u_{(x)}$ by:

$$u_{(x)} = \begin{cases} = u_\infty + \dfrac{u_0 - u_\infty}{\left(1 + \dfrac{x}{4\beta y_0}\right)^2} \;; & \text{in the range } \xi \leq 4; \text{ i.e. } x \leq 4\beta y_0 \\ = u_\infty + \dfrac{u_0 - u_\infty}{x}\beta y_0 \;; & \text{for } \xi \geq 4; \text{ i.e. } x \geq 4\beta y_0 \end{cases} \qquad (43')$$

The enclosed graphs (6,7,8) compare the results for $u_{(x)}$ obtained by means of equations (13-15) for angles θ_1 = 0.025 radians, θ_2 = 0.05 radians and θ_3 = 0.125 radians with those obtained by means of equation (43).

As can be seen, the agreement for all three angles is quite good. Using the method of characteristics, the computation of $u_{(x)}$ takes several hours, while using the equation (43), this time is restricted only to minutes.

18. THE DISTRIBUTION OF PERTURBATION VELOCITY IN THE SPACE AROUND A STRAIGHT LINE DIFFUSER.

Taking into consideration the result obtained in section 14, equation (31), the $\underline{u}$ velocity around the straight line diffuser in the case $\theta \to 0$ may be presented in the following operational form:

$$\frac{u_{(x,y)}}{u_0} = p^{-1}\left[\frac{K_0(\beta y p)}{K_1(\beta y_0 p)}\right] \qquad (44)$$

After considerations similar to those of section 15 it is possible to obtain:

$$\frac{u(x,\bar{y})}{u_0} = \int_0^\infty \frac{K_0(\bar{y}t)\, I_1(t) + K_1(t)\, I_0(\bar{y}t)}{K_1^2(t) + \pi^2 I_1^2(t)} \frac{e^{-\frac{tx}{\beta y_0}}}{t} dt \tag{45}$$

where $\bar{y} = y/y_0$

Expanding the function $\frac{K_0(ap)}{K_1}$ into asymptotic series, one gets:

$$\frac{K_0(ap)}{K_1(p)} \simeq a^{-\frac{1}{2}} e^{-(a-1)p} \left[1 - \frac{1+3a}{8a} p^{-1} + \frac{9+6a+33a^2}{128\, a^2} p^{-2} - \right.$$

$$\left. - \frac{75+27a+33a^2+249a^3}{1024\, a^3} p^{-3} + \ldots \right] \tag{46}$$

Submitting each term of (46) to the transformation $\mathfrak{P}^{-1}$, taking into consideration the "shift effect" and separately the "scale effect", one finally obtains:

$$\frac{u(\bar{x},\bar{y})}{u_0} \simeq \bar{y}^{-\frac{1}{2}} \left\{ 1 - \frac{3+\bar{y}^{-1}}{8} \left[\frac{\bar{x}}{\beta y_0}\right] + \frac{33+6\bar{y}^{-1}+9\bar{y}^{-2}}{256} \left[\frac{\bar{x}}{\beta y_0}\right]^2 - \right.$$

$$\left. - \frac{83+11\bar{y}^{-1}+9\bar{y}^{-2}+25\bar{y}^{-3}}{2048} \left[\frac{\bar{x}}{\beta y_0}\right]^3 + \ldots \right\} \tag{47}$$

where $\bar{x} = x - \beta(y - y_0)$

As can be seen from figure 10, the variable $\bar{x}$ has a very simple geometrical meaning. It represents, namely, the distance, on the horizontal axis, between the point $A(x, y)$ and the first Mach line. It is possible therefore, to consider it as the shifted x co-ordinate of the point. It is obvious that for the diffuser surface the variable $\bar{x}$ reduces itself to the variable x.

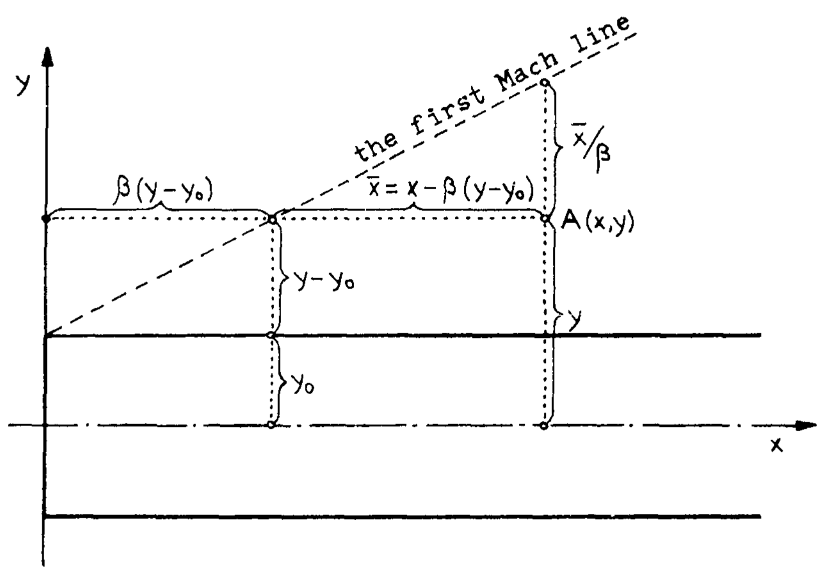


Fig. 10

It is evident that all points on the same Mach line (see figure 10) have the same value for their co-ordinates $\bar{x}$; especially, for the first Mach line, it is $\bar{x} = 0$, and therefore, formula (47) gives immediately:

$$\frac{u(0,\bar{y})}{u_0} = \bar{y}^{-\frac{1}{2}} \tag{48}$$

result which was already obtained, for a more general case, in section 5, equation (8).

It is convenient, by analogy with the function $\sigma(\xi)$, to define a function:

$$\sigma(\bar{\xi},\bar{y}) = \frac{u(\bar{\xi},\bar{y})}{u(0,\bar{y})} \tag{49}$$

where $\bar{\xi} = \frac{\bar{x}}{\beta y_0}$, and $\bar{y}$ is treated as a parameter. This function is, of course, a generalization of the function $\sigma(\xi)$ and may be treated as a limit of the function (compare equation 16):

$$\sigma(\bar{x},\theta,\beta,\bar{y}) = \frac{u(\bar{x},\bar{y}) - u_\infty}{u(0,\bar{y}) - u_\infty}$$

when $\theta \rightarrow 0$, and $\bar{y}$ = const.

From (47) and (48) one gets immediately:

$$\sigma(\bar{\xi},\bar{y}) = 1 - \frac{3+\bar{y}^{-1}}{8}\bar{\xi} + \frac{33+6\bar{y}^{-1}+9\bar{y}^{-2}}{256}\bar{\xi}^2 - \ldots \tag{50}$$

As one should expect, in the limiting case $\bar{y}$ = 1, the variable $\bar{\xi}$ reduces itself to ξ , and we have

$$\sigma(\bar{\xi},1) = \sigma(\xi)$$

which can be verified, comparing the formulae (50) and (40)

For $\bar{y} = \infty$ we have a second limiting case. For small values of $\bar{\xi}$ the function $\sigma(\bar{\xi},\infty)$ is determined by the series:

$$\sigma(\bar{\xi},\infty) = 1 - \frac{3}{8}\bar{\xi} + \frac{33}{256}\bar{\xi}^2 - \frac{83}{2048}\bar{\xi}^3 + \dots \tag{51}$$

and for arbitrary $\bar{\xi}$:

$$\sigma(\bar{\xi},\infty) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{K_1(t)}{K_1^2(t) + \pi^2 I_1^2(t)} \frac{e^{-\bar{\xi}t}}{t^{3/2}}\, dt \tag{51'}$$

The shape of the functions $\sigma(\bar{\xi},1)$ and $\sigma(\bar{\xi},\infty)$ does not differ essentially. Both functions have the same initial values (=1) and both approach zero, for large values of $\bar{\xi}$, as $\bar{\xi}^{-1}$. The slope of $\sigma(\bar{\xi},1)$ at the point $\bar{\xi} = 0$ is $-\frac{1}{2}$ and that of $\sigma(\bar{\xi},\infty)$ is $-\frac{3}{8}$

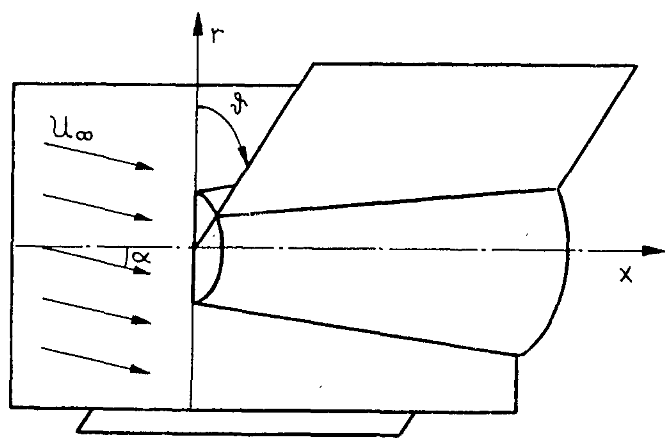
The functions $\sigma(\bar{\xi},\bar{y})$ for $\bar{y}$ = const. differ still less from the function $\sigma(\bar{\xi},1) = \sigma(\xi)$, because their values are situated between these two limiting cases (i.e. $\bar{y} = 1$ and $\bar{y} = \infty$). The functions $\sigma(\bar{\xi},1) = \sigma(\xi)$, $\sigma(\bar{\xi},2)$ and $\sigma(\bar{\xi},\infty)$, for values of $\bar{\xi}$ in the range between 0 and 1, are shown on the graph 9 .

19. THE ANGLE OF INCIDENCE

In the case of axial flow without an angle of incidence, the angle between the flow direction and the diffuser surface is constant for a given cross section. However, for the flow with an angle of incidence α this is no more the case. For this reason it is convenient now to assume a cylindrical co-ordinate system x, r, ϑ such that

(i) - the $\underline{x}$ axis coincides with the axis of the diffuser

(ii) - the polar angle ϑ is chosen in such a manner that the meridian plane which contains both the direction of the undisturbed velocity, and the $\underline{x}$ axis corresponds to the value $\vartheta = 0$ (see figure 11).



<u>Fig. 11</u>

It is obvious that the angle of incidence $\alpha_{(\vartheta)}$ of the meridian component of the undisturbed velocity (for a meridian plane with an angle ϑ) is:

$$\tan \alpha_{(\vartheta)} = \tan \alpha \cos \vartheta$$

(see figure 12)

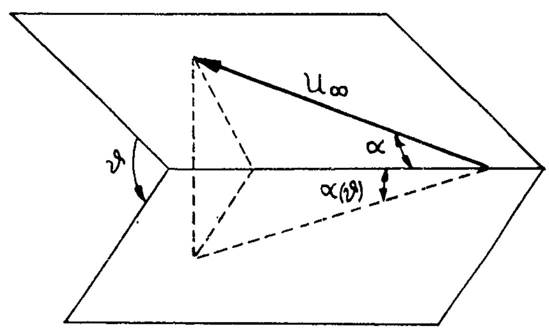


Fig. 12

or, for small values of α :

$$\alpha_{(\vartheta)} \simeq \alpha \cos \vartheta$$

Naturally, the component of the undisturbed velocity u_∞ , normal to the meridian plane, is not disturbed by the diffuser and, consequently, does not influence the potential φ of the distrubed portion of the flow.

As is known from the theory of small disturbances for a body of revolution, the part of the potential φ which is generated by the disturbances, in the case of a flow with an angle of incidence, can be decomposed into two potentials:

1) the first one, φ_1 , equal to the corresponding potential of the flow without an angle of incidence, and

2) the second one, or cross flow potential,

$$\varphi_2 = A \frac{\partial \varphi}{\partial r} \cos \vartheta$$

where A is to be obtained from the boundary condition, namely:

$$\frac{\partial \varphi_2}{\partial r} = - U_\infty \alpha \cos \vartheta$$

Taking into consideration that the operational form of φ_1 is already known (see section 14) and can be represented as

$$P[\varphi_1] = B \; K_0 (\beta r p)$$

(where B does not depend on r), the following result can be immediately obtained:

$$P[\varphi_2] = P[A \frac{\partial \varphi_1}{\partial r} \cos \vartheta] = A \cos \vartheta \frac{\partial}{\partial r} P[\varphi_1] =$$

$$= C \beta p \cos \vartheta \; K_1(\beta r p)$$

where C is the value to be obtained from the boundary condition just mentioned. The operational form of this condition is:

$$\mathcal{P}\left[\frac{\partial \varphi_2}{\partial r}\right] = \frac{\partial}{\partial r}\mathcal{P}[\varphi_2] =$$

$$= \frac{1}{2} C \beta^2 p^2 \cos\vartheta \left[K_0(\beta r p) + K_2(\beta r p)\right] =$$

$$= -\alpha U_\infty \cos\vartheta$$

The value of C, resulting from the last formula, is

$$C = \frac{2\alpha U_\infty}{\beta^2 p^2 [K_0(\beta r p) + K_2(\beta r p)]}$$

and therefore:

$$\mathcal{P}[\varphi_2] = -\frac{2\alpha U_\infty \cos\vartheta \; K_1(\beta r p)}{\beta p [K_0(\beta r p) + K_2(\beta r p)]} \tag{52}$$

Recalling that:

$$u_2(x) = \frac{\partial \varphi_2}{\partial x}$$

and therefore:

$$\mathcal{P}\left[u_{2(x)}\right] = \mathcal{P}\left[\frac{\partial \varphi_2}{\partial x}\right] = p \cdot \mathcal{P}\left[\varphi_2\right]$$

one obtains finally:

$$\frac{u_{2(x)}}{\alpha U_\infty} = \mathcal{P}^{-1}\left[-\frac{2K_1(\beta r p)\cos\vartheta}{\beta\left[K_0(\beta r p) + K_2(\beta r p)\right]}\right] \tag{53}$$

which is the operational representation of the additional u_2 velocity generated by the angle of incidence of the stream.

The non-operational form of $u_{2(x)}$ can be obtained by a procedure analogous to that of section 15. It may be performed by changing, in a suitable way, the path of integration of the integral involved in the operation $\mathcal{P}^{-1}$ or by approximate methods, such as finding the asymptotic expansions of the Bessel functions K_0, K_1 and K_2 for large values of $\underline{p}$

It is also possible to extend these results to the whole space around the diffuser, and for a non-limiting case of $\theta = 0$, by use of the same methods and considerations analogous to those made in order to generalize the results of the case without an incidence angle, as shown in the former sections.

The only difference in comparison with the former methods will appear because of the singular points of the functions $K_1(z)$ and $K_2(z)$. In view of these singular points, it is necessary to take into consideration the residues of the integrand, while changing the path of integration in order to perform the operation $\mathfrak{P}^{-1}$ in the formula (53).

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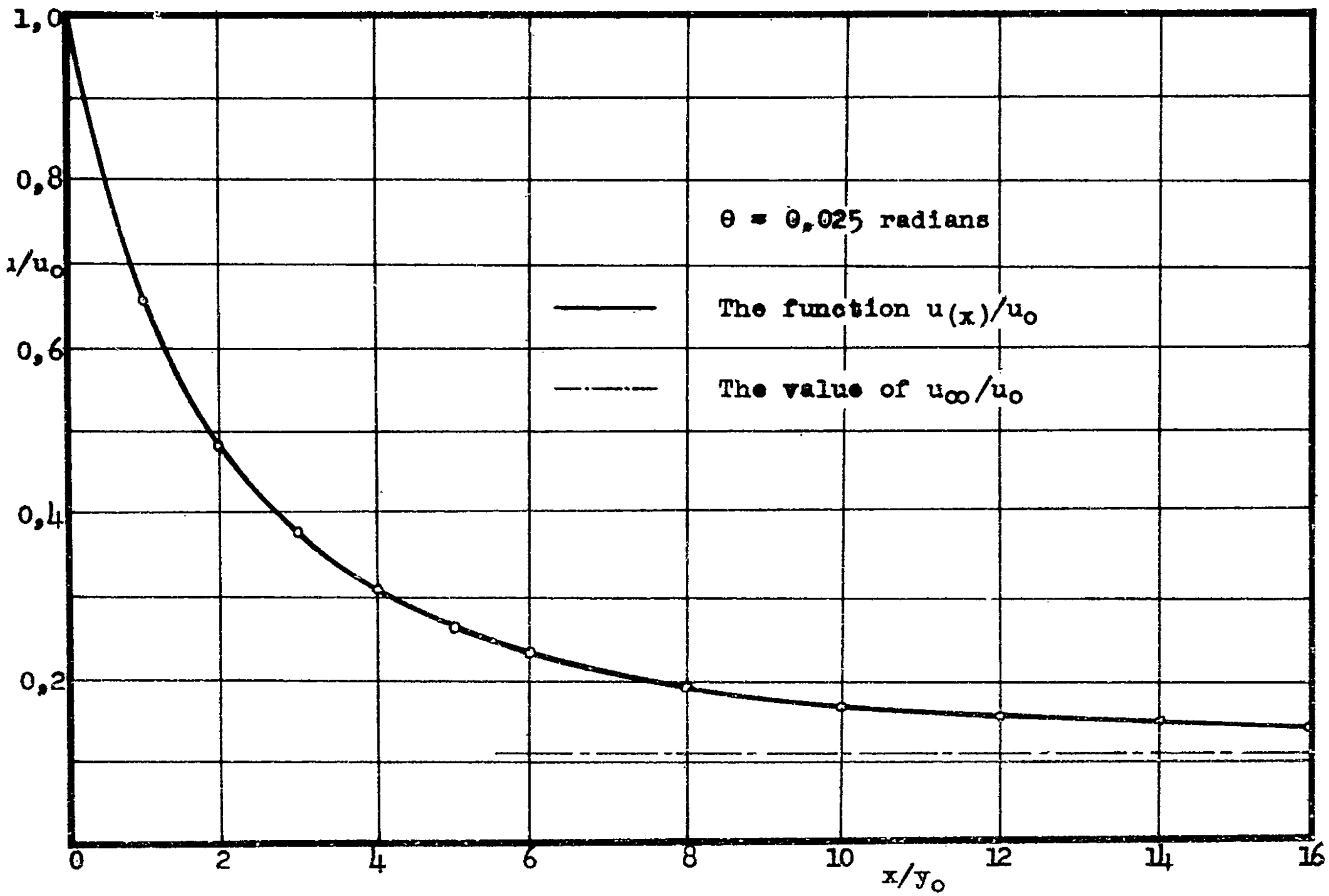
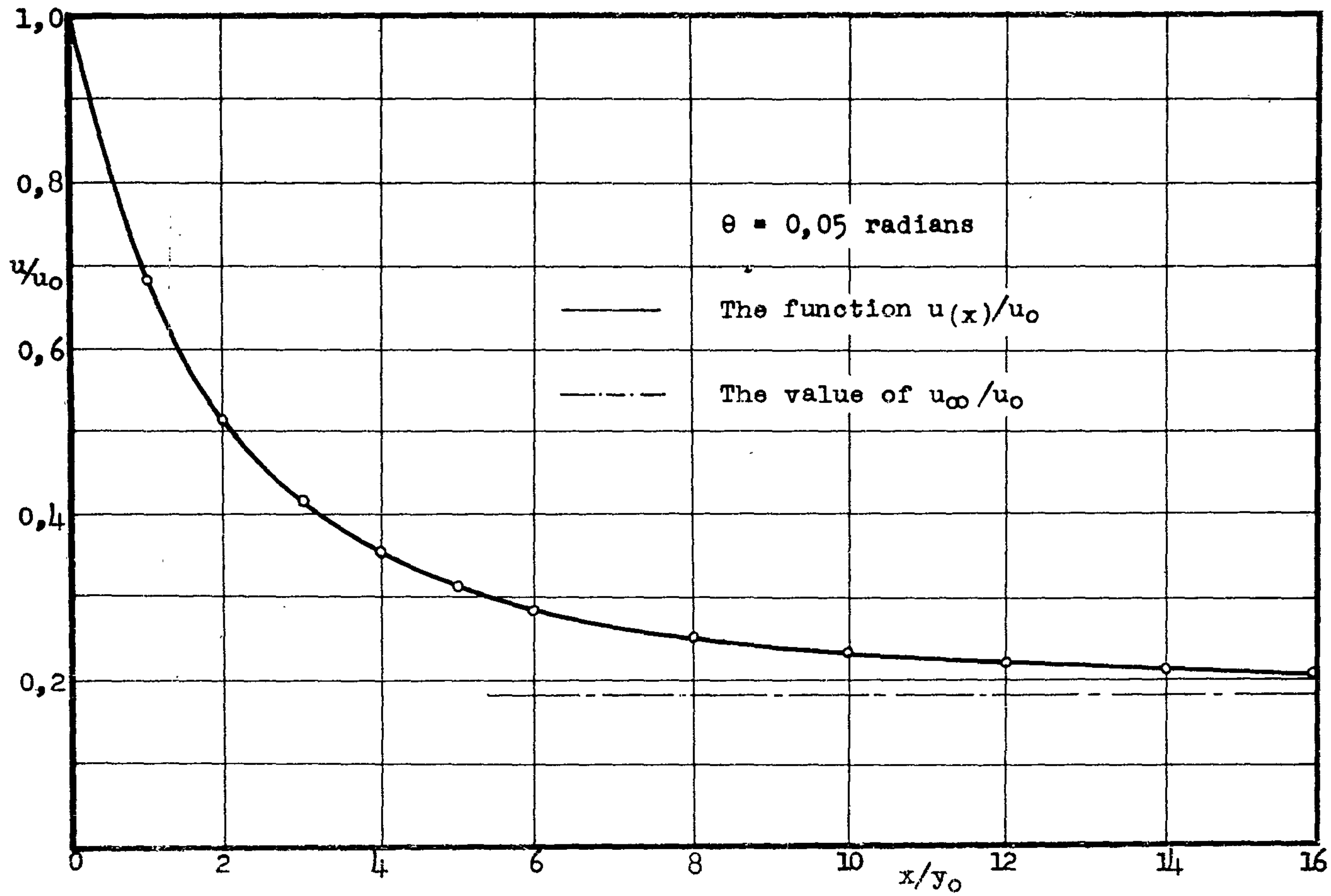
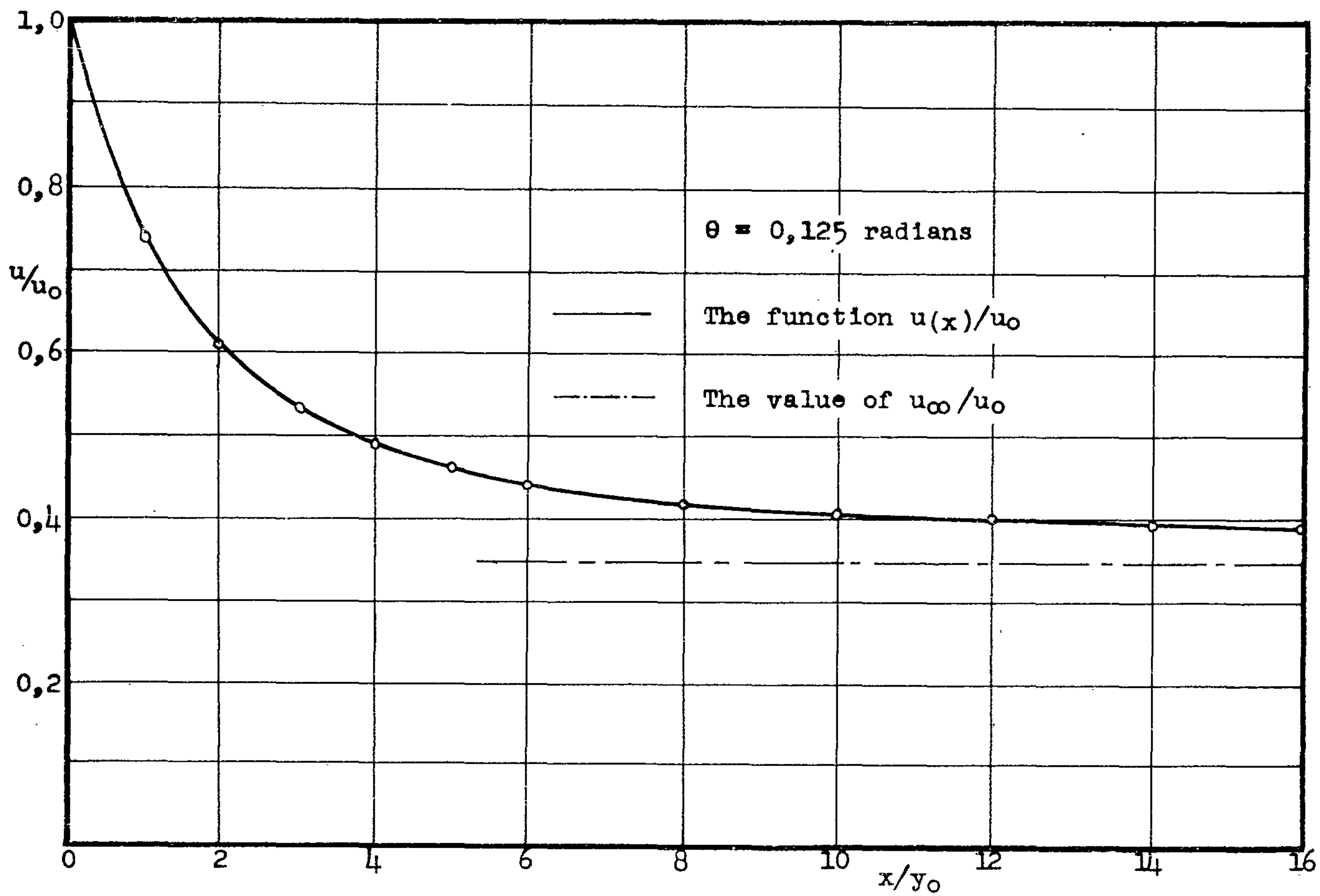


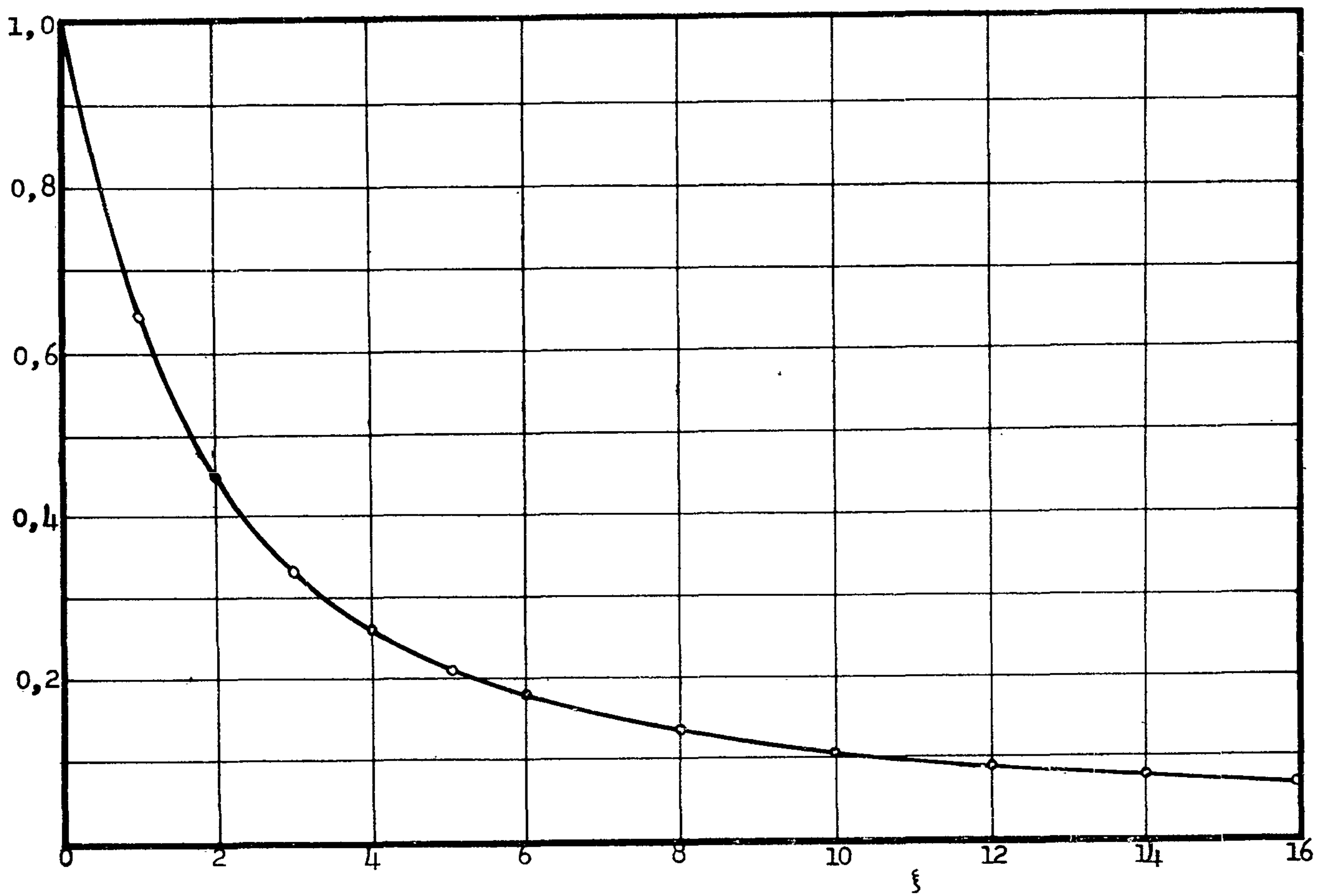
Graph 1 - u-velocity along a straight line diffuser - characteristics method.



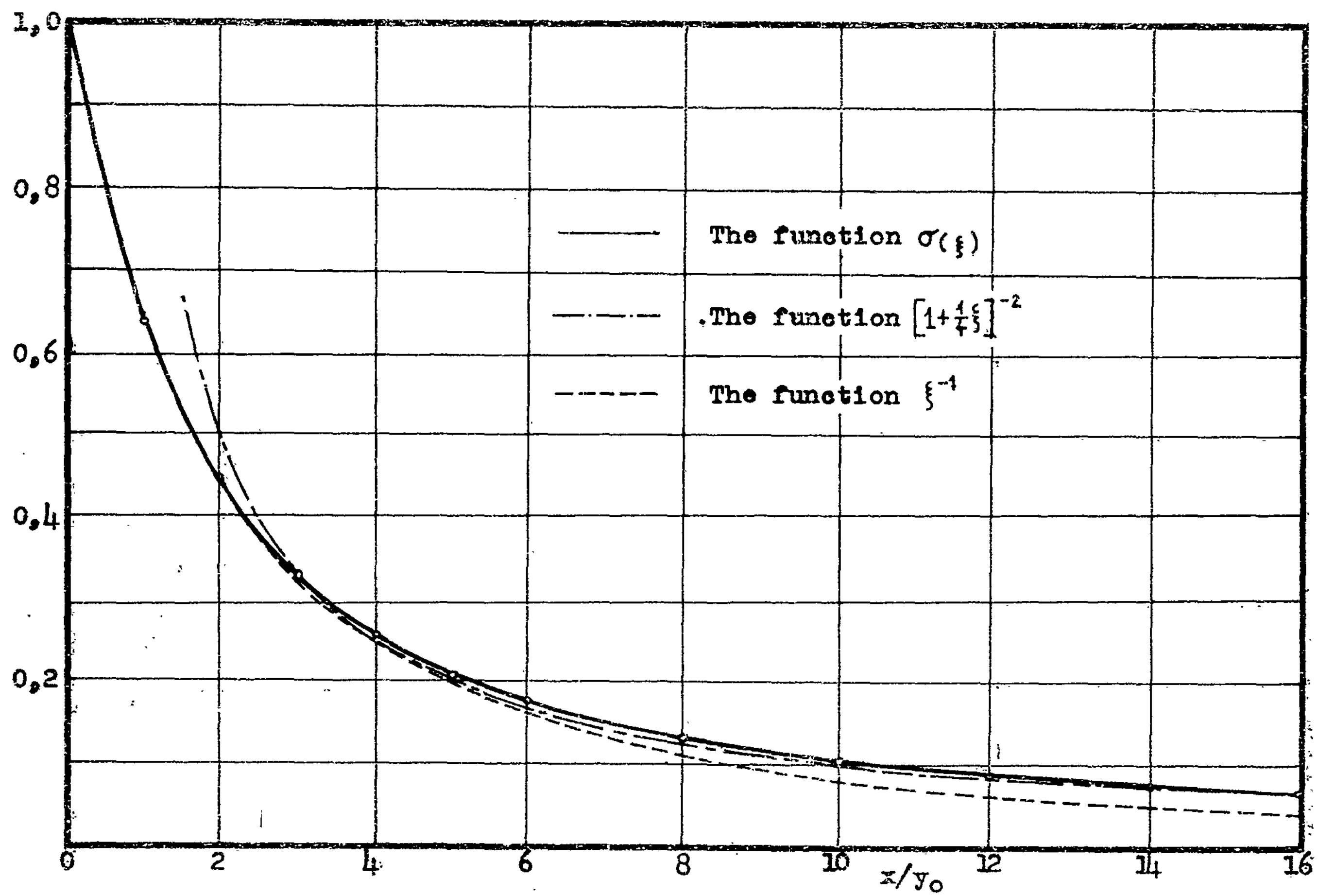
Graph 2 - u-velocity along a straight line diffuser - characteristics method.



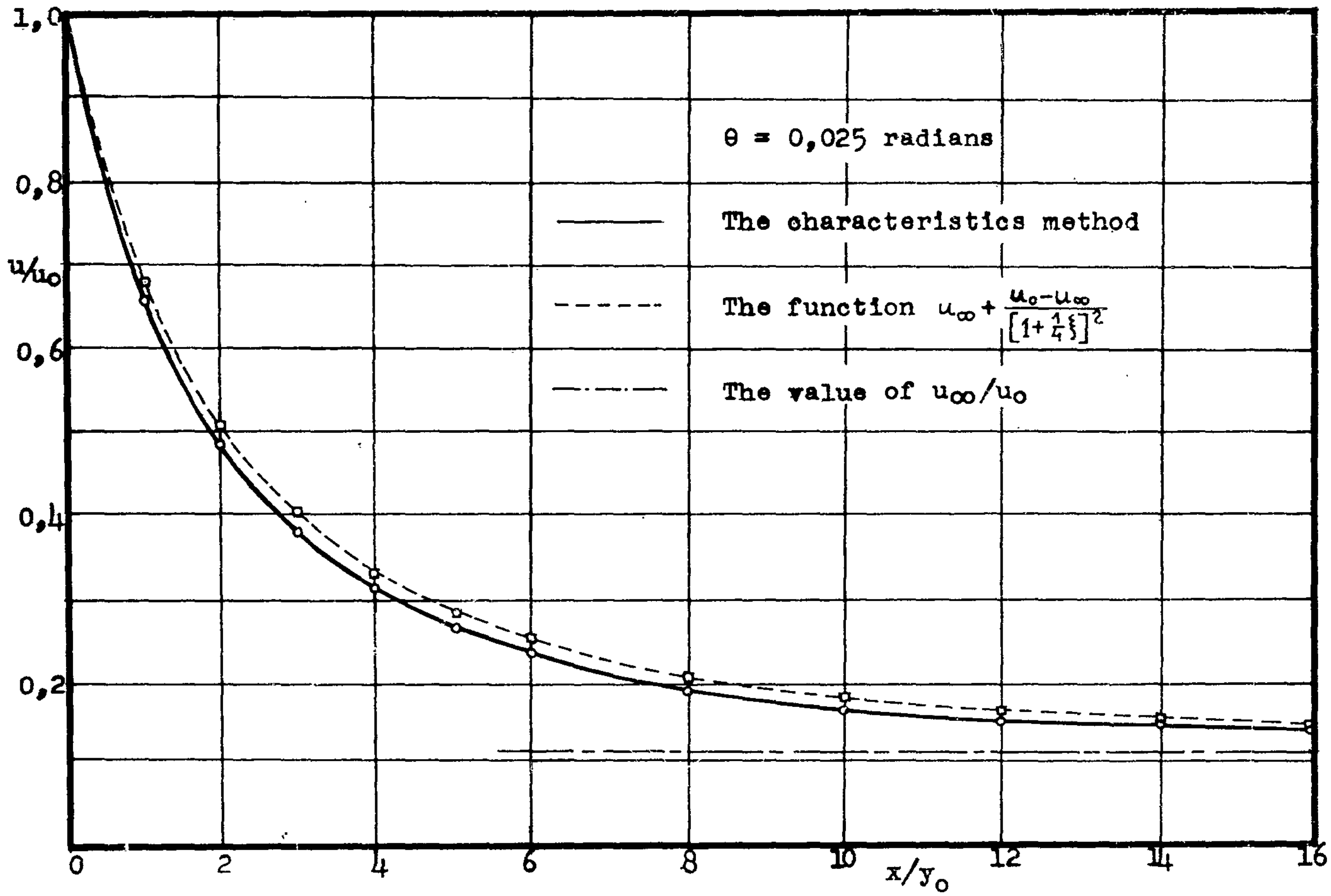
Graph 3 - u-valocity along a straight line diffuser - characteristics method.



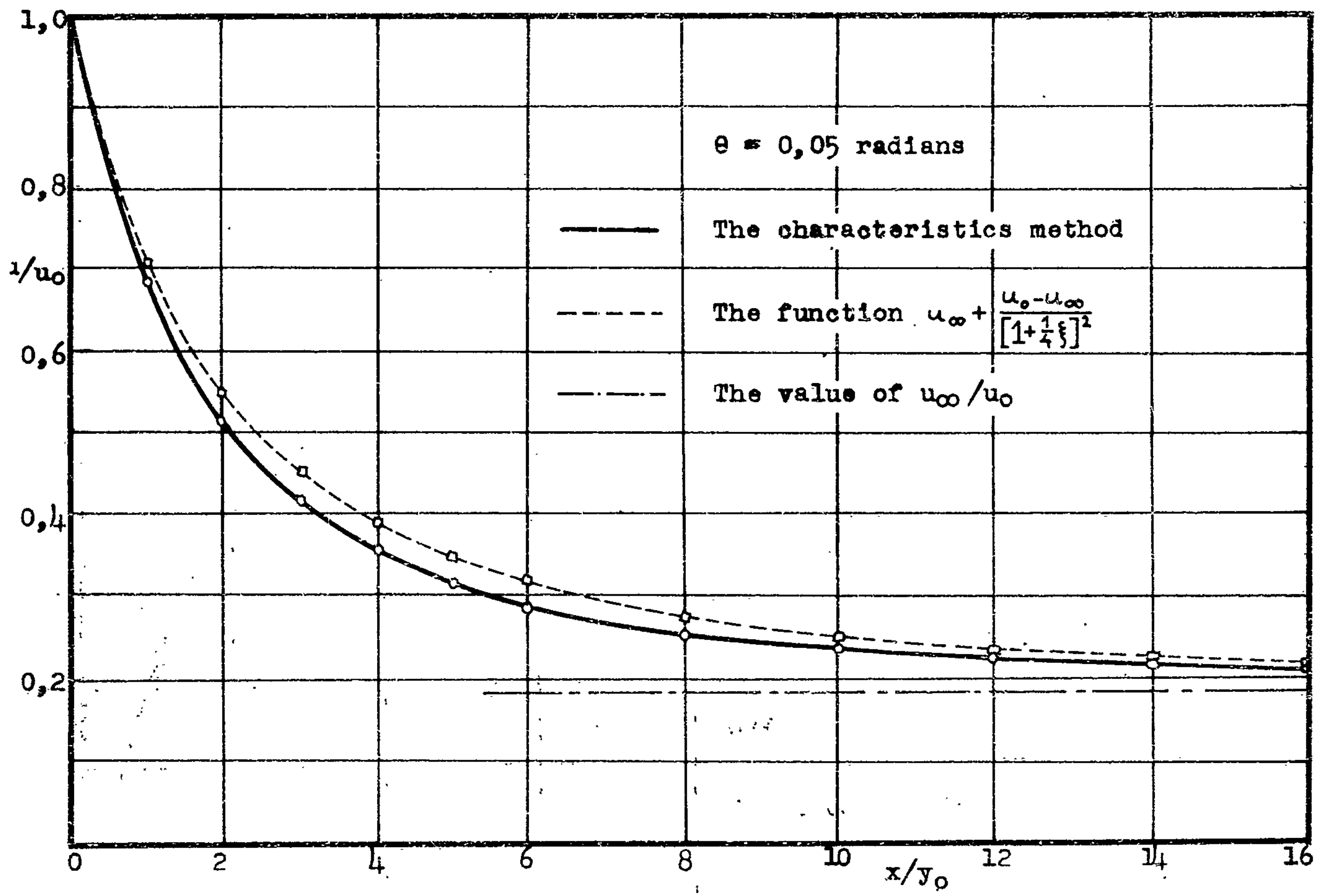
Graph 4 - Function $\sigma(\xi)$



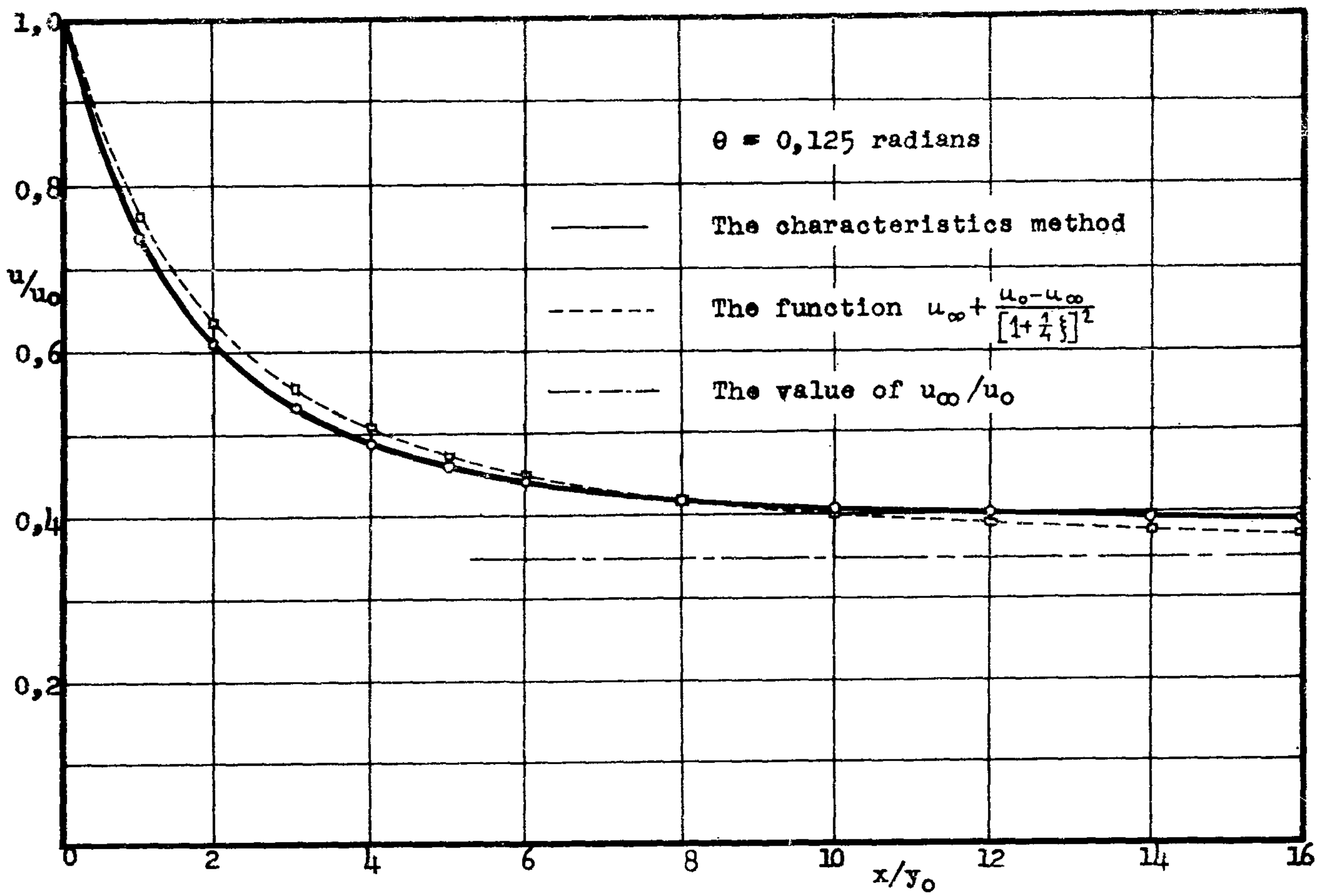
Graph 5 - Comparison between the function $\sigma(\xi)$ and its approximations.



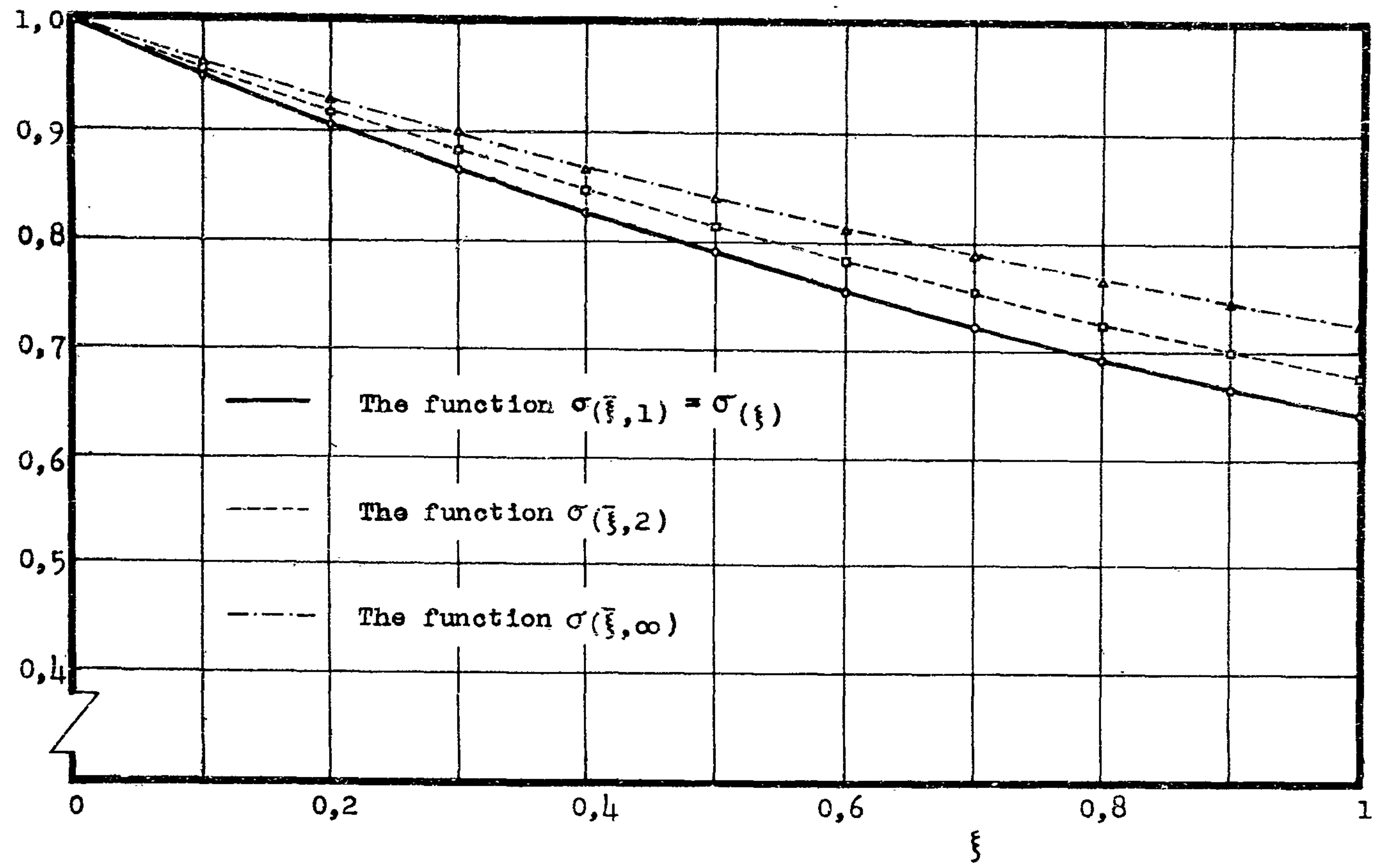
Graph 6 - Comparison between the characteristics method and the analytical solution.



Graph 7 - Comparison between the characteristics method and the analytical solution.



Graph 8 - Comparison between the characteristics method and the analytical solution.



Graph 9 - Comparison between the functions $\sigma(\bar{\xi},1) = \sigma_{(\xi)}$; $\sigma(\bar{\xi},2)$ and $\sigma(\bar{\xi},\infty)$.